ON AMITSUR'S COMPLEX[1]

BY

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1. **Introduction.** Given a commutative ring C and a commutative C-algebra F, Amitsur [3] introduced the following complex: Let

$$F^k = F\otimes_C F\otimes_C \cdots \otimes_C F \quad (k \text{ factors})$$

and F^{k*} the group of units of F^k. Define ring homomorphisms ϵ_i $(i=1, \cdots, k+1)$ of F^k to F^{k+1} by[2,3] $\epsilon_i(f_1\otimes \cdots \otimes f_k)=f_1\otimes \cdots \otimes f_{i-1}\otimes 1\otimes f_i \cdots \otimes f_k$ and define a multiplicative homomorphism Δ_k: $F^{k*}\to F^{(k+1)*}$ by $\Delta_k(x) = \epsilon_1(x)(\epsilon_2(x))^{-1} \cdots (\epsilon_{k+1}(x))^{\pm 1}$. Then the groups F^{k*} and mappings Δ_k form the Amitsur complex; the kth homology group $\operatorname{Ker}\Delta_{k+1}/\operatorname{Im}\Delta_k$ we denote by $H^k(F)$.

In case F is a finite dimensional extension field, Amitsur showed that $H^2(F)$ is isomorphic to the Brauer group of central simple C-algebras split by F. He also showed that in case F is a normal separable extension, $H^n(F)$ is isomorphic to $H^n(G, F^*)$ the nth cohomology group of the Galois group G of F over C with coefficients in the group of nonzero elements of F.

In this paper, we extend and simplify Amitsur's results. We begin by showing (§2) that in case F is a separable field extension of C, $H^n(F) \cong H^n([G:H], K^*)$, where K is a normal closure of F with Galois group G, H is the subgroup corresponding to F, and the cohomology group on the right side is the relative cohomology group as introduced in [1]. Next, we study $H^n(F)$ when C is not necessarily a field but when $n=2$. In §3, under weak hypotheses on F, we exhibit a homomorphism of $H^2(F)$ to the (generalized) Brauer group of central separable algebra classes split by F [5]. This homomorphism is an isomorphism under stronger hypotheses on F and C. These hypotheses are slightly weaker than assuming all projective C, F, and F^2 modules are free and include the cases (1°) C is semilocal (not necessarily Noetherian) and F is a C-algebra which is a finitely generated projective C-module containing $C\cdot 1$ as a direct summand and (2°) $C=K[x]$, $F=L[x]$, with K a field and L a finite dimensional commutative K-algebra.

Hochschild in [13] has given a description of the Brauer group in case F is a purely inseparable extension field of C of exponent 1. In [3, §7], Amitsur

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(2) Henceforth all tensor product signs without subscripts will denote tensor products over C.

(3) Strictly there should be a suffix k on the ϵ_i and in §3 on the η_i. We omit this suffix and leave it to the reader to determine from context which F^k is the domain of the ϵ_i at hand. This is especially true in equations (3.3) and their applications.

tried to show that in this case $H^2(F)$ is isomorphic to the group given in [13]. However (see §4), his proof contains two irreparable errors. By different methods, we exhibit an isomorphism of $H^2(F)$ to the group introduced by Hochschild.

Finally (§5), we apply our methods to get an alternate proof of a theorem of Auslander and Goldman [5]: the natural mapping of the full Brauer group of $K[x]$ into the full Brauer group of the field K (induced by setting $x=0$) is a monomorphism if and only if K is perfect.

2. **Separable field extensions.** Let C be a field, F an extension field of finite degree over C. We pick a fixed normal(4) closure, K, of F over C and denote by Φ_n the set of all n-tuples $\phi=(\phi_1, \phi_2, \cdots, \phi_n)$ of C-algebra monomorphisms of F into K. Furthermore, we let G be the group of all automorphisms of K over C. Since for any g in G the composite $g\phi_i$ will be a monomorphism of F to K whenever ϕ_i is, we define an operation of G on Φ_n by

$$g(\phi_1, \phi_2, \cdots, \phi_n) = (g\phi_1, g\phi_2, \cdots, g\phi_n).$$

Next, we define a pairing of F^n with Φ_n to K, namely a function p on $F^n \times \Phi_n$ to K, C-linear in its first variable and satisfying

$$p(a_1 \otimes a_2 \cdots \otimes a_n, (\phi_1, \phi_2, \cdots, \phi_n)) = \phi_1(a_1)\phi_2(a_2) \cdots \phi_n(a_n).$$

For fixed ϕ in Φ_n and fixed x in F^n, respectively, let p_ϕ and p_x denote the partial maps of F^n to K and Φ_n to K defined by $p_\phi(x)=p(x, \phi)$ and $p_x(\phi)=p(x, \phi)$. Then p_x is homogeneous in the sense that

$$p_x(g\phi) = gp_x(\phi) \qquad \text{for all } \phi \text{ in } \Phi_n, \text{ all } g \text{ in } G, \tag{2.1}$$

and p_ϕ is actually a C-algebra homomorphism of F^n into K. Since K is a field finite over C, Ker p_ϕ is a maximal ideal in F^n.

LEMMA 2.1. *For every maximal ideal M in F^n, there is a $\phi\in\Phi_n$ such that $M=$* Ker p_ϕ. *Two elements ϕ and ϕ' in Φ_n are such that* Ker $p_\phi=$ Ker $p_{\phi'}$ *if and only if $\phi=g\phi'$ for some g in G.*

Proof. There are n embeddings σ_i of F into F^n given by $\sigma_i(a)=1\otimes \cdots \otimes 1\otimes a\otimes 1 \cdots \otimes 1$ (a in the ith spot, $i=1, \cdots, n$). If M is any maximal ideal in F^n then F^n/M is a field generated by the n-isomorphic images $\pi\sigma_i(F)$ of F, where π is the natural homomorphism $F^n \to F^n/M$. Thus any embedding of F^n/M into the algebraic closure of K will automatically fall into K. Hence there is a monomorphism θ of F^n/M into K. If we let ϕ be the n-tuple $(\theta\pi\sigma_1, \cdots, \theta\pi\sigma_n)$ then $\theta\pi$ and p_ϕ coincide on each $\sigma_i(F)$ and so coincide on all of F^n. Thus $M=\text{Ker }\theta\pi=\text{Ker }p_\phi$.

If Ker $p_\phi=$ Ker $p_{\phi'}$ then there is an isomorphism g_0 of Im p_ϕ onto Im $p_{\phi'}$ such that $p_{\phi'}$ is the composite g_0p_ϕ. Extending g_0 to an automorphism g of K, [19, p. 75] we have, for every x in F^n,

(4) Normal is used here in the sense of [19], i.e., we do not asume that K is separable over C.

$$p(x, \phi') = gp(x, \phi) = p(x, g\phi).$$

Setting $x=\sigma_i(a)$, $\phi=(\phi_1, \cdots, \phi_n)$, $\phi'=(\phi_1', \cdots, \phi_n')$, we see that $\phi_i'(a) = g\phi_i(a)$ for all i and all a in F, and so $\phi_i' = g\phi_i$, $\phi' = g\phi$. The reverse implication is clear.

Let R_n be the ring of all mappings $f\colon \Phi_n \to K$ satisfying $f(g\phi) = gf(\phi)$ for all g in G and all ϕ in Φ_n. Then p_x is in R_n for any x in F^n. Our next lemma asserts that the p_x range over R_n with no repetition if F is separable over C, so that p is a "dual" pairing in a certain sense.

LEMMA 2.2. *The mapping $x \to p_x$ of F^n into R_n is a C-algebra homomorphism whose kernel is the radical of F^n. If F is separable over C, this mapping is an isomorphism.*

Proof. The condition $p_x=0$ is equivalent to $p(x, \phi)=0$ for all ϕ, i.e., $x \in \bigcap \operatorname{Ker} p_\phi$. By Lemma 2.1, this means that x is in the intersection of all the maximal ideals of F^n.

If F is separable over C then F^n is semisimple and $x \to p_x$ is a monomorphism. Now if $f \in R_n$ then we proceed to construct x in F^n with $f=p_x$: for fixed $\phi=(\phi_1, \cdots, \phi_n)$, $p(F^n, \phi)$ is clearly the subfield $L = \prod_i \phi_i(F)$ of K. But $f(\phi)$ is also in L; for if we choose g to be any automorphism of K over L then $g\phi_i(a) = \phi_i(a)$ for all a in F, hence $g\phi = \phi$ and $gf(\phi) = f(g\phi) = f(\phi)$; thus $f(\phi)$ is left fixed by all the automorphisms of K over L so $f(\phi)$ is in L. Therefore we conclude that for each ϕ there exists an element $x(\phi) \in F^n$ such that $f(\phi) = p(x(\phi), \phi)$. To produce a single x with $f=p_x$, we must find an x so that $x \equiv x(\phi)$ (mod Ker p_ϕ) for every ϕ. Since the Ker p_ϕ are maximal ideals (not necessarily distinct), the Chinese Remainder Theorem asserts that these congruences are solvable provided $x(\phi) \equiv x(\phi')$ (mod Ker p_ϕ) whenever Ker p_ϕ = Ker $p_{\phi'}$. But this consistency condition is verified because, by Lemma 2.1, $\phi' = g\phi$ and so

$$p(x(\phi), \phi') = p(x(\phi), \phi) = gp(x(\phi), \phi) = gf(\phi) = f(g\phi) = f(\phi') = p(x(\phi'), \phi')$$

so that $x(\phi) - x(\phi') \in \operatorname{Ker} p_{\phi'}$. This proves Lemma 2.2.

In [1], Adamson defined the following cochain complex(5):

$$C^{n-1} = \text{the set of } f \text{ in } R_n \text{ such that } f(\phi) \neq 0 \text{ for all } \phi$$
$$= \text{the multiplicative group of units in } R_n,$$

$\delta\colon C^{n-1} \to C^n$ is defined by $\delta f(\phi) = \prod f(\phi_1, \cdots, \hat{\phi}_i, \cdots, \phi_{n+1})^{(-1)^{i-1}}$, where, as usual, the circumflex means omission. Comparing with Amitsur's complex, we clearly have for every x in F^n

(5) If H is the subgroup of G leaving F elementwise fixed, the elements of Adamson's C^{n-1} are functions of n-tuples of left cosets in G/H. But there is a one-to-one correspondence between left cosets in G/H and monomorphisms of F into K.

$$p(\epsilon_i x, (\phi_1, \phi_2, \cdots, \phi_{n+1})) = p(x, (\phi_1, \phi_2, \cdots, \hat{\phi}_i, \cdots, \phi_{n+1}))$$

and $p_x \in C^{n-1}$ whenever $x \in F^{n*}$. This implies

$$p(\Delta_n x, (\phi_1, \cdots, \phi_{n+1})) = \delta p_x(\phi_1, \cdots, \phi_{n+1})$$

so that $x \to p_x$ sends F^{n*} into C^{n-1} and is a homomorphism of complexes. If F is separable over C, Lemma 2.2 shows that $x \to p_x$ is actually an isomorphism of complexes.

The cohomology groups of Adamson's complex $\{C^{n-1}, \delta\}$ are called the relative cohomology groups(5) of G modulo H with coefficients in K^*, and are written $H^n([G:H], K^*)$. In [1], it is also shown that if F is separable and normal over C then $H^n([G:H], K^*) = H^n(G, F^*)$, the usual cohomology group of G with coefficients in F^*. Thus we sum up our results in

THEOREM 1. *Let C be a field and F an extension field of finite degree over C. Then there is a homomorphism of the Amitsur complex of F into the Adamson complex of F over C inducing a homomorphism of $H^n(F)$ into $H^n([G:H], K^*)$. If F is separable over C these homomorphisms are isomorphisms. In particular, if F is normal and separable over C then $H^n(F)$ is isomorphic to $H^n(G, F^*)$.*

REMARK. By [8, Satz 7] and [1, Theorem 9.2], for general separable fields F, $H^2(F)$ has now been proved to be isomorphic to the Brauer group of central simple C-algebras split by F.

The result of Theorem 1 in the case when F is normal separable is obtained in [3, Theorem 6.1] by more complicated computations. It is worth noting that in this case the mapping $x \to p_x$ is just an isomorphism of the Amitsur complex $\{F^{n*}\}$ with the homogeneous cochain complex used to define the cohomology groups of G in [11].

3. **The Brauer group.** Let C be any commutative ring with unit. We say a C-algebra is *split* if it is isomorphic to $\mathrm{End}_C(V)$(6) for some finitely generated faithful projective C-module V. If F is a commutative C-algebra a C-algebra A is said to be *split by F* if the F-algebra $A \otimes F$ is split.

LEMMA 3.1. *Let F be a commutative C-algebra which is a flat C-module*(7) *and such that the mapping $c \to c \cdot 1$ of C to F is a split monomorphism*(8). *Then*

(a) *if B is a C-algebra then B is central separable*(9) *if and only if $B \otimes F$ is a central separable F-algebra,*

(b) *every C-algebra split by F is central separable.*

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(6) We shall use the notation $\mathrm{End}_R(M)$ for the R-endomorphism ring of an R-module M, i. e. $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$.

(7) I.e., whenever $X \to Y$ is a monomorphism of C-modules, then so is $X \otimes F \to Y \otimes F$ (cf. [9, p. 122]).

(8) I.e., there is a C-module mapping $\phi: F \to C$ such that $\phi(c \cdot 1) = c$ for $c \in C$.

(9) B is central if the center of B is $C \cdot 1$ and is isomorphic to C. As in [5], B is separable, if B is projective as a $B \otimes B^0$-module, where B^0 is the opposite algebra of B. When C is a field "central separable" is equivalent to "central simple and finite dimensional."

Proof. Since a split F-algebra is a central separable F-algebra [5, Proposition 5.1], the second statement will follow from the first. Now, if B is separable, B is a direct summand in a free $B\otimes B^0$-module. Tensoring with F, we have $B\otimes F$ projective as a module over $B\otimes B^0\otimes F \cong (B\otimes F)\otimes_F(B\otimes F)^0$, so $B\otimes F$ is a separable F-algebra. For the converse, we use another definition of projectivity of the $B\otimes B^0$-module B: every epimorphism of $B\otimes B^0$-modules $X\to B$ splits. If $\alpha: X\to B$ is an epimorphism then $\alpha\otimes 1: X\otimes F\to B\otimes F$ is also an epimorphism of $B\otimes B^0\otimes F$-modules which splits because $B\otimes F$ is projective, giving a mapping $\beta: B\otimes F\to X\otimes F$ with $(\alpha\otimes 1)\beta$ the identity. By hypothesis if $i: C\to F$ is defined by $i(c)=c\cdot 1$, we have a mapping $\phi: F\to C$ with $\phi i=$ the identity. Consider the induced mappings $1\otimes i: B=B\otimes C\to B\otimes F$ and $1\otimes\phi: X\otimes F\to X\otimes C=X$. The composite $(1\otimes\phi)\beta(1\otimes i)$ is the required splitting mapping $B\to X$, since $\alpha[(1\otimes\phi)\beta(1\otimes i)]=(1\otimes\phi)(\alpha\otimes 1)\beta(1\otimes i)$ $=(1\otimes\phi)(1\otimes i)=$ the identity.

To prove $B\otimes F$ is central if and only if B is, notice that the center of B is $\mathrm{End}_{B\otimes B^0}(B)$ and the center of $B\otimes F$ is $\mathrm{End}_{(B\otimes F)\otimes_F(B\otimes F)^0}(B\otimes F)$ $=\mathrm{End}_{(B\otimes B^0)\otimes F}(B\otimes F)\cong \mathrm{End}_{B\otimes B^0}(B)\otimes F$, by the assertion "$\phi_3$ is an isomorphism" in [9, p. 210]. In fact, we have shown that if B is separable, the center of $B\otimes F$ is the tensor product of F with the center of B.

The *full Brauer group* $\mathcal{B}(C)$ is defined following [5, §5; 7, p. 132]: the elements are equivalence classes of central separable C-algebras under the equivalence relation: $A\sim B$ if and only if $A\otimes E\cong B\otimes E'$ for some split algebras E, E'. The group operation is induced by tensor product. For the proof that this gives a group whose identity element is exactly the class of split algebras, see [5, Theorem 5, Proposition 5.3]. We shall be concerned primarily with the *Brauer group of F over C* denoted $\mathcal{B}(F/C)$ and defined as the subgroup of $\mathcal{B}(C)$ consisting of those equivalence classes which consist of algebras split by F(10). To verify that this is indeed a subgroup, we note first that if A and B are C-algebras split by F, so is $A\otimes B$, since $(A\otimes B)\otimes F$ $\cong(A\otimes F)\otimes_F(B\otimes F)$ which is split because it is the tensor product of split algebras; and the opposite A^0 is split by F because $A^0\otimes F\cong(A\otimes F)^0$ which is also split [5, Proposition 5,1, 5.3 and Corollary 5.4].

The algebras we study are primarily algebras of endomorphisms of F^2 and F^3. However, particularly in the proof of Theorem 3, we meet more general representations on $V\otimes F$ and $V\otimes F^2$ where V is a C-module not necessarily F. To study these we must introduce some notations.

If $x\in F^n$ we use $L(x)$ for the endomorphism of F^n produced by multiplication by x. We shall also use the notation $L(1\otimes x)$ for the operator $1\otimes L(x)$ on $V\otimes F^n$ (so that when $V=F$ the two L-notations will coincide).

We extend the ϵ's to mappings on $\mathrm{End}_C(V\otimes F^n)$ as follows: By the commutativity and associativity of the tensor product, we have isomorphisms $\pi_i: V\otimes F^n\to V\otimes F^n$ $(i=2,\cdots,n+1)$ induced by the cyclic permutation

(10) Using [5, Propositions 5.1 and 5.3] it can easily be verified that if one algebra in a class is split by F then all algebras in that class are split by F.

$(i, \cdots, n+1)$ that puts the final F in the ith place (counting the first place as occupied by V):

$$\pi_i(v \otimes f_2 \cdots \otimes f_{n+1}) = v \otimes f_2 \otimes \cdots \otimes f_{i-1} \otimes f_{n+1} \otimes f_i \otimes f_{i+1} \cdots \otimes f_n.$$

To each e in $\mathrm{End}_C(V\otimes F^{n-1})$ corresponds the endomorphism $e\otimes 1$ of

$$V\otimes F^{n-1}\otimes F = V\otimes F^n$$

which in turn is carried by π_i to(3)

$$\eta_i e = \pi_i(e \otimes 1)\pi_i^{-1} \qquad (i = 2, \cdots, n+1). \tag{3.1}$$

Thus η_i is a C-algebra homomorphism of $\mathrm{End}_C(V\otimes F^{n-1})$ into $\mathrm{End}_C(V\otimes F^n)$ for $i=2, \cdots, n+1$.

More explicitly, $\eta_i e$ is the endomorphism of $V\otimes F^n$ such that, for all $v\in V$ and $f_j\in F$,

$$(\eta_i e)(v \otimes f_2 \otimes \cdots \otimes f_{n+1})$$
$$= L(1 \otimes \cdots \otimes f_i \otimes \cdots \otimes 1)\epsilon_i'\left[e(v \otimes f_2 \otimes \cdots \otimes \hat{f}_i \otimes \cdots \otimes f_{n+1})\right]$$

where ϵ_i' $(i>1)$ is defined on $V\otimes F^{n-1}$ by inserting 1 in the ith place.

Analogously, we define η_1: $\mathrm{End}_C(F^n)\rightarrow\mathrm{End}_C(V\otimes F^n)$ by $\eta_1 e=1\otimes e$, or

$$(\eta_1 e)(v \otimes x) = v \otimes e(x) \qquad \text{for } v \in V,\ x \in F^n. \tag{3.2}$$

In short, $\eta_i e$ is the identity on the ith factor of $V\otimes F^n = V\otimes F\otimes \cdots \otimes F$ and acts on the other factors the way e does.

Note that $\eta_1 L(F^n) = L(1\otimes F^n)$ is the set of scalar operators on the F^n-module $V\otimes F^n$.

The principal property of these η's (both when $V=F$ and in general) is the set of relations among composites (as for face operators in a semisimplicial complex)

$$\eta_i\eta_j = \eta_{j+1}\eta_i \qquad \text{for } i \leqq j. \tag{3.3}$$

These same identities hold for the ϵ's in the Amitsur complex and in fact are the reasons that it is a complex, i.e., that $\Delta_n\Delta_{n-1}=1$.

From the definition, it is clear that if $V=F$

$$\eta_i L(x) = L(\epsilon_i x), \qquad x \in F^n. \tag{3.4}$$

We denote by E_{n+1} the centralizer $\mathrm{End}_{1\otimes F^n}(V\otimes F^n)$ of $\eta_1 L(F^n)$ in $\mathrm{End}_C(V\otimes F^n)$. In particular, $E_1=\mathrm{End}_C(V)$. Again from the definition of η_i it is clear that

$$\eta_i E_n \subset E_{n+1}, \qquad i = 2, \cdots, n+1. \tag{3.5}$$

In the rest of this section, we assume that *V and F are finitely generated faithful projective C-modules* and that *the unit mapping*(8) *$C\rightarrow F$ is a split monomorphism.* If we assume only that F is a finitely generated projective

C-module and that the unit mapping is a monomorphism then it automatically splits (and F is a faithful C-module). This can be seen by replacing [9, VII, Example 11] by [9, VI, Example 11] in the proof of [4, Lemma 4.7].

We draw four consequences of these hypotheses:

(3.6) *Each η_i is a monomorphism*

because $\eta_i e=0$ in (3.1) implies $e\otimes 1=0$; but the projection $F\to C$, which results from the splitting of the unit mapping, will map $\operatorname{Im}(e\otimes 1)$ onto $\operatorname{Im} e$. Hence $e=0$.

For every subset $S=\{i, j, k, \cdots\}$ of $\{1, 2, \cdots, n\}$, arranged so that $i>j>k>\cdots$, define the "shuffle" η_S as the composite $\eta_i\eta_j\eta_k\cdots$. If m is the number of elements in S and $1\notin S$ then by (3.5) η_S carries E_{n-m} into E_n. If $1\in S$, then η_S is defined on $\operatorname{End}_C(F^{n-m})$ and, in particular, will carry $L(F^{n-m})$ into E_n.

LEMMA 3.2. *Let S and T be complementary subsets of $\{1, 2, \cdots, n\}$ and let $1\in T$. Then the mapping $E_{n-m}\otimes F^m\to E_n$ defined by $e\otimes x\to(\eta_S e)(\eta_T L(x))$ is an isomorphism of C-algebras. In particular, $E_n=(\eta_S E_{n-m})\eta_T L(F^m)$.*

Proof. In [9, XI, p. 205], a natural mapping ϕ_3: $\operatorname{Hom}_\Lambda(A, B)\otimes\operatorname{Hom}_\Gamma(A', B')\to\operatorname{Hom}_{\Lambda\otimes\Gamma}(A\otimes A', B\otimes B')$ is defined, where Λ and Γ are C-algebras, A and B are left Λ-modules, and A' and B' are left Γ-modules. If A and A' are finitely generated projective modules ϕ_3 is an isomorphism [9, p. 210]. Taking $\Lambda=1\otimes F^{n-m-1}$, $\Gamma=F^m$, $A=B=V\otimes F^{n-m-1}$, $A'=B'=\Gamma$, we have an isomorphism $E_{n-m}\otimes F^m\to E_n$, which is exactly the mapping described in the lemma for $T=\{1, 2, \cdots, n-m\}$. For any other subset of $\{1, 2, \cdots, n\}$, we need only permute the factors F in the preceding isomorphism.

We shall say that two subalgebras A and B of a C-algebra E are in *tensor product relation* if $ab=ba$ for every a in A, b in B, and if the natural C-algebra homomorphism $a\otimes b\to ab$ is an isomorphism of $A\otimes B$ onto $AB\subset E$.

COROLLARY 3.3. *With notations as in Lemma 3.2, if A is a C-subalgebra of $\eta_S E_{n-m}$, then A and $\eta_T L(F^m)$ are in tensor product relation.*

Proof. The natural mapping $A\otimes\eta_T L(F^m)\to E_n$ defined by $x\otimes y\to xy$ can be factored $A\otimes\eta_T L(F^m)\to\eta_S E_{n-m}\otimes\eta_T L(F^m)\to E_n$, where the mapping on the left is a monomorphism, since the C-module $\eta_T L(F^m)\cong F^m$ is flat, and the mapping on the right is an isomorphism, by the lemma and (3.6).

Another important property of the η's is that they can spread algebras out when embedding them in E_n so that they commute:

LEMMA 3.4. *Let $A\subseteq\eta_i E_n$, $i>1$, and $B\subseteq\eta_1\operatorname{End}_{\epsilon_{i-1}F^{n-1}}(F^n)$. Then A and B commute elementwise.*

Proof. By Lemma 3.2 with $S=\{2, \cdots, n\}$ and $T=\{1\}$, $E_n\cong E_1\otimes F^{n-1}$

so that every element of $\eta_i A$ is a sum of elements of the form $e\otimes L(f_2)\otimes\cdots\otimes L(f_{i-1})\otimes 1\otimes L(f_i)\otimes\cdots\otimes L(f_n)$ where $e\in E_1, f_j\in F$, 1 denotes the identity map and $\otimes$ denotes the tensor product of mappings on $V\otimes F\otimes\cdots\otimes F$. By a similar argument, the elements of $\eta_1 B$ can be written as sums of products of the form $1\otimes L(f_2')\otimes\cdots\otimes L(f_{i-1}')\otimes e'\otimes L(f_i')\otimes\cdots\otimes L(f_n')$ with $e'\in \mathrm{End}_C(F)$. But these two products clearly commute.

Amitsur, in part inspired by the work of I. Schur and R. Brauer, associated cocycles of F^{3*} to central simple C-algebras somewhat as follows: let C be a field, A a central simple C-algebra split by an extension field F so that $A\otimes F\cong \mathrm{End}_F(W)$. This W can be written as $V\otimes F$ with V a C-space. Thus A is mapped into $E_2=\mathrm{End}_{1\otimes F}(V\otimes F)$. Composing this mapping with $\eta_2, \eta_3: E_2\to E_3$, one obtains two mappings of A into E_3. By judicious application of a generalized Skolem-Noether theorem, these can be shown to be carried into each other by an inner automorphism of E_3. This automorphism is generated by an element P which must satisfy a kind of cocycle identity: $(\eta_4 P)(\eta_2 P)(\eta_3 P^{-1})=\eta_1 L(t)=L(\epsilon_1 t)$ for some t in F^{3*}. Amitsur showed that the correspondence $A\to t$ induces an isomorphism $\mathfrak{B}(F/C)\to H^2(F)$.

This program can be extended to the case of certain rings C and F but with comparatively strong hypotheses (see the epimorphism proof in Theorem 3). Therefore we prefer to treat the inverse mapping, which requires only the hypotheses we have already imposed on C and F and associates to a cocycle t in F^{3*} a central separable algebra with a particularly good representation (Lemma 3.5). Thus we obtain a homomorphism $H^2(F)\to\mathfrak{B}(F/C)$ and reserve the stronger hypotheses for the proof that this is an isomorphism.

Let P be an invertible element in $E_3=\mathrm{End}_{1\otimes F^2}(V\otimes F^2)$ and define

$$(3.7)\qquad A(P)=\{a\in E_2=\mathrm{End}_{1\otimes F}(V\otimes F)\mid P(\eta_2 a)P^{-1}=\eta_3 a\}.$$

When $V=F$ and $P=L(t)$ for some $t\in F^{3*}$, we write $A(t)$ for $A(L(t))$:

$$(3.7')\qquad A(t)=\{a\in \mathrm{End}_{1\otimes F}(F^2)\mid L(t)(\eta_2 a)L(t)^{-1}=\eta_3 a\}.$$

LEMMA 3.5. *If $t\in F^{3*}$ then $A(t)$ contains $L(F\otimes 1)$ as a maximal commutative subalgebra.*

Proof. That $L(F\otimes 1)=\eta_2 L(F)\subset A(t)$ follows directly from the commutativity of $L(F^2)$ and (3.3). If $a\in A(t)$ and a commutes with $L(F\otimes 1)$ then a commutes with $L(F\otimes 1)L(1\otimes F)=L(F^2)$ so that a is of the form $L(s)$ with $s=\sum f_i\otimes f_i'\in F\otimes F$. The defining identity for $A(t)$ then implies $\eta_3 a=\eta_2 a$, or $\sum f_i\otimes f_i'\otimes 1=\sum f_i\otimes 1\otimes f_i'$. Applying the contraction $f_1\otimes f_2\otimes f_3\to f_1 f_2\otimes f_3$ of F^3 to F^2 we have $\sum f_i f_i'\otimes 1=\sum f_i\otimes f_i'$ so that $s\in F\otimes 1$, $a\in L(F\otimes 1)$.

LEMMA 3.6. *$A(P)$ and $\eta_1 L(F)=L(1\otimes F)$ are in tensor product relation in E_2.*

Proof. Since $\eta_1 L(F)$ is in the center of E_2, it commutes with $A(P)$ and we have a homomorphism $A(P)\otimes F\to A(P)(\eta_1 L(F))$. We must show that if $\sum a_i\eta_1 L(f_i)=0$ with $a_i\in A(P), f_i\in F$, then $\sum a_i\otimes f_i=0$ in $A(P)\otimes F$. We have

$$0 = P(\eta_2[\sum a_i\eta_1 L(f_i)])P^{-1} = \sum (\eta_3 a_i)\eta_2\eta_1 L(f_i)$$

since $a_i \in A(P)$ and P commutes with $\eta_2\eta_1 L(F)$. But $\eta_3 A(P) \subset \eta_3 E_2$ and $\eta_3 E_2$ and $\eta_2\eta_1 L(F)$ are in tensor product relation (Corollary 3.3) and so

$$\sum (\eta_3 a_i) \otimes \eta_2\eta_1 L(f_i) = 0.$$

Since η_3 and the mapping $f_i \to \eta_2\eta_1 L(f_i)$ are monomorphisms and F is C-flat, $\sum a_i \otimes f_i = 0$ in $E_2 \otimes F$, and hence also in $A(P) \otimes F$.

LEMMA 3.7. *$A(P)\eta_1 L(F) = E_2$ if and only if $(\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1} = \eta_1 L(u)$ for some $u \in F^{3*}$. This u is necessarily a cocycle, i.e., $\Delta_3 u = 1$. In particular, if t is a cocycle in F^{3*}, $A(t)\eta_1 L(F) = \mathrm{End}_{1\otimes F}(F^2)$.*

Proof. First suppose $A(P)\eta_1 L(F) = E_2$. Apply the three embeddings η_2, η_3, η_4 of E_3 into E_4 to the defining equation for $A(P)$ and obtain for all a in $A(P)$,

$$(\eta_i P)(\eta_i \eta_2 a)(\eta_i P)^{-1} = \eta_i \eta_3 a, \qquad i = 2, 3, 4.$$

Using (3.3), specifically $\eta_2\eta_2 = \eta_3\eta_2$, $\eta_2\eta_3 = \eta_4\eta_2$, $\eta_3\eta_3 = \eta_4\eta_3$, we see that conjugating by $(\eta_3 P)^{-1}$, then by $\eta_2 P$, then by $\eta_4 P$ carries $\eta_4\eta_3 a$ successively to $\eta_3\eta_2 a$, to $\eta_4\eta_2 a$, to $\eta_4\eta_3 a$. Thus $Q = (\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1}$ commutes with $\eta_4\eta_3 A(P)$. By (3.5), $Q \in E_4$, so that Q commutes with

$$\begin{aligned}
\eta_4\eta_3 A(P)\cdot \eta_1 L(F^3) &= (\eta_4\eta_3 A(P))(L(1 \otimes F \otimes 1 \otimes 1))(L(1 \otimes 1 \otimes F \otimes F)) \\
&= \eta_4\eta_3(A(P)\eta_1 L(F))\cdot L(1 \otimes 1 \otimes F \otimes F) \\
&= (\eta_4\eta_3 E_2)\cdot \eta_2\eta_1 L(F^2) \\
&= E_4
\end{aligned}$$

where the last two equalities follow by hypothesis and by Lemma 3.2, respectively. Thus Q is in the center of E_4 which is $\eta_1 L(F^3)$(11).

Conversely, suppose $Q = (\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1}$ is in the center of E_4 and let a be any element of E_2. Then $Q(\eta_4\eta_3 a)Q^{-1} = \eta_4\eta_3 a$. Using $\eta_4\eta_3 = \eta_3\eta_3$, this may be rewritten as

$$(\eta_2 P)\eta_3(P^{-1}(\eta_3 a)P)(\eta_2 P)^{-1} = \eta_4(P^{-1}(\eta_3 a)P)$$

or, if $b = P^{-1}(\eta_3 a)P$, as

$$(3.8) \qquad (\eta_2 P)(\eta_3 b)(\eta_2 P)^{-1} = \eta_4 b.$$

Lemma 3.2 yields the isomorphism $\psi: E_2 \otimes F \to E_3$, given by $\psi(\sum b_i \otimes f_i) = \sum (\eta_2 b_i)(\eta_3\eta_1 L(f))$. Hence we may write $b = \sum (\eta_2 b_i)(\eta_3\eta_1 L(f_i))$ and substitute into (3.8). Using (3.3), specifically $\eta_3\eta_2 = \eta_2\eta_2$, $\eta_3\eta_3 = \eta_4\eta_3$ and $\eta_4\eta_2 = \eta_2\eta_3$, we get

(11) If V is free over C then the ring of endomorphisms of $V \otimes F^3$ over $1 \otimes F^3$ is a matrix ring whose center is well known to be $\eta_1 L(F^3)$. If V is just faithful and projective, the same result can be proved in a variety of ways (e.g. [6, Proposition A.3 and Theorem A.2(g); 16, Lemma 3.3]).

$$\sum \eta_2(P(\eta_2 b_i)P^{-1})\eta_4\eta_3\eta_1 L(f_i) = \sum \eta_2(\eta_3 b_i)\eta_4\eta_3\eta_1 L(f_i).$$

Another application of Lemma 3.2 and Corollary 3.3 shows that this means

$$\sum P(\eta_2 b_i)P^{-1} \otimes f_i = \sum (\eta_3 b_i) \otimes f_i \qquad \text{in } E_3 \otimes F.$$

Thus $\sum b_i \otimes f_i$ is in the kernel of $\phi \otimes 1: E_2 \otimes F \to E_3 \otimes F$ where $\phi: E_2 \to E_3$ is defined by $\phi(x) = P(\eta_2 x)P^{-1} - \eta_3 x$. Since F is C-flat, $\text{Ker}(\phi \otimes 1) = \text{Ker}\, \phi \otimes F = A(P) \otimes F$ by (3.7). Hence $\sum b_i \otimes f_i = \sum b_i' \otimes f_i'$ with $b_i' \in A(P), f_i' \in F$. Returning to b and a, we have

$$\begin{aligned}\eta_3 a = PbP^{-1} &= P\psi(\sum b_i' \otimes f_i')P^{-1} = \sum P\eta_2 b_i' P^{-1}\eta_3\eta_1 L(f_i') \\ &= \sum \eta_3 b_i' \eta_3\eta_1 L(f_i') \in \eta_3(A(P)\eta_1 L(F)).\end{aligned}$$

Since η_3 is a monomorphism, by (3.6), $a \in A(P)\eta_1 L(F)$.

It remains to show that if $(\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1} = \eta_1 L(u)$, then $\Delta_3 u = 1$. In general, the same identities that show $\Delta_3\Delta_2 = 1$, show, even without commutativity, that if $x = (\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1}$ then $(\eta_5 x)(\eta_3 x)(\eta_4 x)^{-1}(\eta_2 x)^{-1} = (\eta_4\eta_4 P)(\eta_2 x)(\eta_4\eta_4 P)^{-1}(\eta_2 x)^{-1}$. In our case, $\eta_2 x = \eta_2\eta_1 L(u)$ commutes with all $\eta_i\eta_j P$ $(i, j > 1)$ and so in fact $1 = (\eta_5 x)(\eta_3 x)(\eta_4 x)^{-1}(\eta_2 x)^{-1} = \eta_1 L(\Delta_3 u)$, by use of (3.3) and (3.4), so that $\Delta_3 u = 1$.

Corollary 3.8. *If $(\eta_4 P)(\eta_2 P)(\eta_3 P^{-1}) = \eta_1 L(u)$ for some $u \in F^{3*}$, then $A(P)$ is a central separable C-algebra split by F. In particular, if t is a cocycle, i.e., $\Delta_3 t = 1$, $A(t)$ is a central separable C-algebra split by F.*

Proof. By Lemma 3.7 $A(P)\eta_1 L(F) = E_2$ and so by Lemma 3.6, $A(P) \otimes F \cong E_2$. Since $E_2 = \text{End}_{1\otimes F}(V \otimes F)$ is a split F-algebra, Lemma 3.1 completes the proof.

Thus we have a mapping from 2-cocycles t in the Amitsur complex to algebra classes in $\mathfrak{B}(F/C)$. Before proving the lemma which will show that this map is a homomorphism we need a preliminary step.

Lemma 3.9. *Let A, B be two central separable C-subalgebras of a C-algebra W which commute elementwise. Then A and B are in tensor product relation in W. If the centrealizer of AB in W is C, then $AB = W$.*

Proof. Since A and B commute, there is a canonical epimorphism $A \otimes B \to AB$. But by [5, Proposition 1.4] $A \otimes B$ is again central separable and so all its ideals are of the form $\mathfrak{c}(A \otimes B)$ with $\mathfrak{c}$ an ideal in C [5, Corollary 3.2]. Hence if the above epimorphism had a nonzero kernel then AB, and so A, would not be faithful C-modules. But A is a faithful C-module [5, Theorem 2.1] and thus $A \otimes B \cong AB$ so that AB is a central separable subalgebra of W. The last statment of the lemma is then an immediate consequence of [5, Theorem 3.3].

Lemma 3.10. *Let P be a unit of E_3 with $(\eta_4 P)(\eta_2 P)(\eta_3 P^{-1}) = \eta_1 L(u)$ for some u in F^{3*} and let t be a cocycle in F^{3*}. Then $A(P) \otimes A(t) \cong A(ut) \otimes E_1$ so that*

$A(P)\otimes A(t)\sim A(ut)$. In particular, if u is a cocycle in F^{3*}, $A(u)\otimes A(t) \cong A(ut)\otimes A(1)$, where $A(1)=\eta_2 \operatorname{End}_C(F)$, and $A(u)\otimes A(t)\sim A(ut)$.

REMARK. In this lemma, we are using simultaneously subsets of $\operatorname{End}_{1\otimes F^n}(V\otimes F^n)$ (like $A(P)$) and subsets of $\operatorname{End}_{1\otimes F^n}(F^{n+1})$ (like $A(t)$), so that our previous definitions and lemmas apply in two ways. To avoid confusion, we shall reserve the notations E_{n+1} for $\operatorname{End}_{1\otimes F^n}(V\otimes F^n)$ and η_i for the corresponding mappings $E_n\to E_{n+1}$. We shall write the other endomorphism rings out explicitly: $\operatorname{End}_{1\otimes F^n}(F^{n+1})$ and shall use η_i' for the corresponding mappings $\operatorname{End}_{1\otimes F^{n-1}}(F^n)\to \operatorname{End}_{1\otimes F^n}(F^{n-1})$. Since the last sentence of the lemma follows directly from the first two, the ultimate confusion of taking $V=F$ in our computation never arises.

Proof. Let W be the C-algebra given by

$$W = \left\{x \in \operatorname{End}_{1\otimes 1\otimes F}(V\otimes F^2) \,\middle|\, (\eta_1 L(t)\eta_2 P)\eta_3 x(\eta_1 L(t)\eta_2 P)^{-1} = \eta_4 x\right\}.$$

We shall show that $W=A_1B_1=A_2B_2$, where A_i and B_i are C-subalgebras in tensor product relation ($i=1, 2$) and $A_1\cong A(P)$, $B_1\cong A(t)$, $A_2\cong A(ut)$, $B_2\cong E_1$.

First, we let $A_1=\eta_2 A(P)$ and $B_1=\eta_1 A(t)$, and we show A_1 and B_1 are contained in W. Since $A(P)\subset E_2$, $\eta_3\eta_2 A(P)$ and $\eta_2 P$ are both in E_4, by (3.5). Hence they commute with $\eta_1 L(F^3)$. Thus, for $a\in A(P)$, using $\eta_3\eta_2=\eta_2\eta_2$, $\eta_2\eta_3=\eta_4\eta_2$ from (3.3) and using the definition (3.7) of $A(P)$,

$$(\eta_1 L(t)\eta_2 P)\eta_3\eta_2 a(\eta_1 L(t)\eta_2 P)^{-1} = (\eta_2 P)(\eta_2\eta_2 a)(\eta_2 P)^{-1} = \eta_2(\eta_3 a) = \eta_4\eta_2 a.$$

Thus $A_1\subset W$. Similarly, by Lemma 3.4, $\eta_2 E_3$ commutes with $\eta_1 \operatorname{End}_{1\otimes F^2}(F^3)$; the former contains $\eta_2 P$ and the latter contains $\eta_1\eta_2' A(t)=\eta_3\eta_1 A(t)$, so that, for $a'\in A(t)$,

$$(\eta_1 L(t)\eta_2 P)\eta_3\eta_1 a'(\eta_1 L(t)\eta_2 P)^{-1} = \eta_1(L(t)\eta_2' a' L(t)^{-1}) = \eta_1\eta_3' a' = \eta_4\eta_1 a'$$

and $B_1\subset W$.

Using Lemma 3.4 again, with $n=2$, $i=2$, we conclude that A_1 and B_1 commute. By Lemma 3.9 and Corollary 3.8, they are in tensor product relation. With Lemma 3.9 in mind, we compute the centralizer of A_1B_1 in W. If w is an element of this centralizer then w commutes with $\eta_2 A(P)L(1\otimes 1\otimes F) = \eta_2 A(P)\eta_2\eta_1 L(F) = \eta_2 E_2$, by Lemma 3.7; w also commutes with $\eta_1 A(t)L(1\otimes 1\otimes F)=\eta_1 A(t)\eta_1\eta_1' L(F)=\eta_1 \operatorname{End}_{1\otimes F}(F^2)$, again by Lemma 3.7. Furthermore, $w\in W$ implies $Q(\eta_3 w)Q^{-1}=\eta_4 w$, with $Q=\eta_1 L(t)(\eta_2 P)\in E_4$. The following lemma then shows $w\in C$ which proves $A_1B_1=W$.

LEMMA 3.11. *If an element w in $\operatorname{End}_{1\otimes 1\otimes F}(V\otimes F^2)$ commutes elementwise with $\eta_2 E_2$ and $\eta_1 \operatorname{End}_{1\otimes F}(F^2)$ and also satisfies $Q(\eta_3 w)Q^{-1}=\eta_4 w$ with $Q\in E_4$, then $w\in C$.*

Proof. By an appeal to [9, p. 210] as in Lemma 3.2, we have an isomorphism

$$\text{End}_{1\otimes F}(V \otimes F) \dot{\otimes} \text{End}_C(F) \to \text{End}_{1\otimes 1\otimes F}(V \otimes F^2)$$

defined for $e \in \text{End}_{1\otimes F}(V\otimes F)$ and $e' \in \text{End}_C(F)$ by

$$e \otimes e' \to (\eta_2 e)(\eta_3\eta_1 e').$$

But by (3.3) $\eta_3\eta_1 \text{End}_C(F) = \eta_1\eta_2' \text{End}_C(F) \subset \eta_1 \text{End}_{1\otimes F}(F^2)$ so that w commutes with every element of $(\eta_2 E_2)(\eta_3\eta_1 \text{End}_C(F)) = \text{End}_{1\otimes 1\otimes F}(V\otimes F^2)$. Thus w is in the center of the latter ring so that $w = L(1\otimes 1\otimes f)$ for some f in F([11]). Then $\eta_3 w = Q(\eta_3 w)Q^{-1} = \eta_4 w$ implies $1\otimes 1\otimes 1\otimes f = 1\otimes 1\otimes f\otimes 1$. Apply to this the mapping $1\otimes 1\otimes 1\otimes \phi$ where $\phi: F\to C$ is an inverse of the split unit mapping, i.e., $\phi(c\cdot 1) = c$ for $c\in C$. We get

$$1\otimes 1\otimes 1\otimes \phi(f) = 1\otimes 1\otimes f\otimes 1 = 1\otimes 1\otimes 1\otimes f.$$

Since $f \to 1\otimes 1\otimes 1\otimes f$ is a monomorphism by the flatness of F, we have $f = \phi(f) \in C$ and $w\in C$.

Now we resume the proof of Lemma 3.10. Before defining A_2 and B_2, we note that

$$PWP^{-1} = \{y \in \text{End}_{1\otimes 1\otimes F}(V \otimes F^2) \mid \eta_1 L(ut)(\eta_3 y)\eta_1 L(ut)^{-1} = \eta_4 y\}$$

because if $x\in W$ and $y = PxP^{-1}$ then

$$[\eta_1 L(t)(\eta_2 P)(\eta_3 P)^{-1}]\eta_3 y[\eta_1 L(t)(\eta_2 P)(\eta_3 P)^{-1}]^{-1} = (\eta_4 P)^{-1}(\eta_4 y)(\eta_4 P)$$

or, using $(\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1} = \eta_1 L(u)$ and the fact that $\eta_4 P\in E_4$ commutes with $\eta_1 L(F^3)$, $\eta_1 L(ut)\eta_3 y\eta_1 L(ut)^{-1} = \eta_4 y$. Since all steps are reversible, the equality is proved. Now define A_2 and B_2 by

$$PA_2P^{-1} = \eta_1 A(ut), \qquad PB_2P^{-1} = \eta_3\eta_2 E_1.$$

It is sufficient to prove that $\eta_1 A(ut)$ and $\eta_3\eta_2 E_1$ are in PWP^{-1}, in tensor product relation and have a product equal to PWP^{-1}. These facts are proved in much the same way as for A_1 and B_1: If $a\in A(ut)$ then $\eta_3\eta_1 a = \eta_1\eta_2' a$ and $\eta_1 L(ut)(\eta_3\eta_1 a)\eta_1 L(ut)^{-1} = \eta_1\eta_3' a = \eta_4\eta_1 a$, by the definition of $A(ut)$. Thus $\eta_1 A(ut) \subset PWP^{-1}$. If $a'\in E_1$, then $\eta_3\eta_3\eta_2 a' = \eta_4\eta_3\eta_2 a'$, by (3.3), and $\eta_3\eta_3\eta_2 a'\in E_4$ commutes with $\eta_1 L(ut)$. Thus $\eta_3\eta_2 E_1 \subset PWP^{-1}$. Since $\eta_3\eta_2 = \eta_2\eta_2$, $\eta_3\eta_2 E_1 \subset \eta_2 E_2$ which commutes with $\eta_1 A(ut)$, by Lemma 3.4, and hence the two algebras are in tensor product relation, by Lemma 3.9. Thus again it remains to show that the centralizer of $\eta_1 A(ut)\cdot \eta_3\eta_2 E_1$ in PWP^{-1} is C. If w is an element of this centralizer, then w commutes with $\eta_1 A(ut)\eta_1\eta_1' L(F) = \eta_1 \text{End}_{1\otimes F}(F^2)$; w also commutes with $\eta_3\eta_2 E_1\eta_2\eta_1 L(F) = \eta_2(\eta_2 E_1\eta_1 L(F)) = \eta_2 E_2$ by Lemma 3.2. Furthermore, $w\in PWP^{-1}$ implies $Q\eta_3 wQ^{-1} = \eta_4 w$ with $Q = \eta_1 L(ut)\in E_4$. Lemma 3.11 then completes the proof.

Finally, we show that $t\to \{A(t)\}$ actually induces a single valued mapping on cohomology classes in $H^2(F)$:

LEMMA 3.12. *If $s\in F^{2*}$ and $t\in F^{3*}$ then*

$$L(s)A(t)L(s^{-1}) = A(t\cdot \Delta_2 s).$$

Proof. An endomorphism x of F^2 is in $L(s)A(t)L(s^{-1})$ if and only if $x \in \mathrm{End}_{1\otimes F}(F^2)$ and

$$L(t)\eta_2(L(s^{-1})xL(s))L(t)^{-1} = \eta_3(L(s^{-1})xL(s)).$$

In view of (3.4) this last equation can be rearranged to give

$$L(t\cdot \epsilon_3 s\cdot \epsilon_2 s^{-1})(\eta_2 x)L(t\cdot \epsilon_3 s\cdot \epsilon_2 s^{-1})^{-1} = \eta_3 x.$$

Since x commutes with $L(1\otimes F)$, $\eta_3 x$ commutes with $L(1\otimes F^2) = L(\epsilon_1 F^2)$. Hence we may conjugate by $L(\epsilon_1 s)$ to get

$$L(t\cdot \Delta_2 s)\eta_2 x L(t\cdot \Delta_2 s)^{-1} = \eta_3 x.$$

This proves the lemma.

Putting together Corollary 3.8, Lemmas 3.10, 3.12 and 3.5, we have

THEOREM 2. *Let F be a commutative C-algebra, finitely generated and projective as a C-module and such that the unit mapping $C\to F$ splits*[8]. *Then the correspondence $t \to A(t) = \{a \in \mathrm{End}_{1\otimes F}(F^2) \mid L(t)\eta_2 a L(t)^{-1} = \eta_3 a\}$ induces a homomorphism $H^2(F) \to \mathfrak{B}(F/C)$. The algebras $A(t)$ which arise this way from cocycles t all contain $L(F\otimes 1) = \eta_2 L(F)$ as a maximal commutative subalgebra.*

In order to show that the homomorphism in Theorem 2 is an isomorphism of $H^2(F)$ onto $\mathfrak{B}(F/C)$ we must impose further hypotheses. A commutative ring R will be said to satisfy hypothesis (H) if

(H): *R is a finite direct sum of rings R_i, and every finitely generated, faithful, projective R_i-module is free.*

REMARKS. 1. If every faithful, finitely generated, projective R_i-module is free, then R_i is indecomposable, for if $R_i = S\oplus T$ then $R_i \oplus S$ is faithful, finitely generated and projective, but not free. Thus the decomposition $R = \sum_\oplus R_i$ in (H) is the unique decomposition of R into indecomposable rings.

2. If W is a faithful, finitely generated, projective R-module and R satisfies (H) then $W = \sum_\oplus W_i$, where each W_i is a free R_i-module on (say) n_i generators. A necessary and sufficient condition for W to be free over R is that $n_i = n_j$ for all i, j.

LEMMA 3.13. (a) *If C and F satisfy* (H) *and A is a C-algebra split by F, then $A\otimes F \cong \mathrm{End}_{1\otimes F}(V\otimes F)$, with V a faithful, finitely generated, projective C-module.*

(b) *If C satisfies* (H) *and V_1 and V_2 are faithful, finitely generated, projective C-modules such that* $\mathrm{End}_{1\otimes F}(V_1\otimes F)$ *and* $\mathrm{End}_{1\otimes F}(V_2\otimes F)$ *are isomorphic F-algebras, then V_1 and V_2 are isomorphic.*

(c) *If a commutative ring R satisfies* (H) *and W is a faithful, finitely generated, projective R-module, then every R-algebra automorphism of* $\mathrm{End}_R(W)$ *is inner.*

Proof. In (a) we are given $A \otimes F = \mathrm{End}_F(W)$ with W a faithful, finitely generated, projective F-module. We are to prove $W = V \otimes F$ for some faithful, finitely generated, projective C-module V. When C decomposes as in (H), all the terms in this hypothesis and conclusion decompose correspondingly. Hence we may assume that C already satisfies the condition that every faithful, finitely generated projective module is free. Since A is central separable, it is a faithful, finitely generated, projective C-module [5, Theorem 2.1] and so now is free. Hence $\mathrm{End}_F(W) \cong A \otimes F$ is a free F-module. On the other hand, if we decompose F into $\sum_{\oplus} F_i$ according to (H), then by Remark 2 above, W decomposes into $\sum_{\oplus} W_i$, with W_i free on n_i generators over F_i, and $\mathrm{End}_F(W) = \sum_{\oplus} \mathrm{End}_{F_i}(W_i)$, where the matrix algebra $\mathrm{End}_{F_i}(W_i)$ is free on n_i^2 generators over F_i. By Remark 2, $\mathrm{End}_F(W)$ is free over F if and only if $n_i^2 = n_j^2$ for all i, j, which implies $n_i = n_j$, which gives W free on n_i generators over F. Thus $W = V \otimes F$, with V free on n_i generators over C.

(b) As in the proof of (a), we immediately may reduce to the case where all faithful, finitely generated, projective C-modules are free. Let V_i be free on n_i generators ($i = 1, 2$). Then $\mathrm{End}_{1 \otimes F}(V_i \otimes F)$ is a free F-module on n_i^2 generators, and the algebra isomorphism is an isomorphism of F-modules. Thus $n_1 = n_2$ and $V_1 \cong V_2$.

(c) When R decomposes as in (H), $\mathrm{End}_R(W)$ decomposes into $\sum_{\oplus} \mathrm{End}_{R_i}(W_i)$, and any automorphism of $\mathrm{End}_R(W)$ induces automorphisms on the direct summands. But R_i certainly satisfies the hypothesis of [5, Theorem 3.6] so that every automorphism of $\mathrm{End}_{R_i}(W_i)$ is inner, whence (c) follows immediately.

THEOREM 3. *Let F be a C-algebra which is finitely generated and projective as a C-module and such that the unit mapping*[8] *$C \to F$ splits. If C, F and $F \otimes F$ satisfy* (H) *then the homomorphism $H^2(F) \to \mathfrak{B}(F/C)$ defined in Theorem 2 is an isomorphism.*

Proof. Let $\{A(t)\} = \{1\}$, that is, $A(t) = \mathrm{End}_C(V)$, with V a faithful, finitely generated projective C-module. Then we have the following F-algebra isomorphisms, by Lemmas 3.2, 3.6 and 3.7: $\mathrm{End}_{1 \otimes F}(V \otimes F) \cong \mathrm{End}_C(V) \otimes F \cong A(t) \otimes F \cong A(t)\eta_1 L(F) = \mathrm{End}_{1 \otimes F}(F^2)$. Lemma 3.13(b) then shows that V is isomorphic to F as a C-module, so that there is a C-algebra isomorphism $\alpha: A(t) \to \mathrm{End}_C(F)$. But, as noted above, $A(t) \otimes F \cong \mathrm{End}_{1 \otimes F}(F^2) \cong \mathrm{End}_C(F) \otimes F$. Hence $\alpha \otimes 1: A(t) \otimes F \to \mathrm{End}_C(F) \otimes F$ induces a $(1 \otimes F)$-algebra automorphism of $\mathrm{End}_{1 \otimes F}(F^2)$. By Lemma 3.13(c), this automorphism is inner, say by an element Q. Then, for every a in $A(t)$,

$$QaQ^{-1} \in \eta_2 \, \mathrm{End}_C(F).$$

Since $\eta_2\eta_2 = \eta_3\eta_2$, we have

$$(\eta_2 Q)(\eta_2 a)(\eta_2 Q)^{-1} = (\eta_3 Q)(\eta_3 a)(\eta_3 Q)^{-1} = (\eta_3 Q) L(t)(\eta_2 a) L(t)^{-1}(\eta_3 Q)^{-1}.$$

Hence $L(t)^{-1}(\eta_3Q)^{-1}(\eta_2Q)$ is an element of $\mathrm{End}_{1\otimes F^2}(F^3)$ commuting with all elements of $\eta_2A(t)$. But

$$\begin{aligned}\eta_2A(t)L(1\otimes F^2)\\ &= (\eta_2A(t)L(1\otimes 1\otimes F))L(1\otimes F\otimes 1) = \eta_2(A(t)L(1\otimes F))L(1\otimes F\otimes 1)\\ &= \eta_2(\mathrm{End}_{1\otimes F}(F^2))\eta_3\eta_1L(F) = \mathrm{End}_{1\otimes F^2}(F^3), \text{ by Lemmas 3.7 and 3.2.}\end{aligned}$$

Thus $L(t)^{-1}(\eta_3Q)^{-1}(\eta_2Q)$ commutes with all of $\mathrm{End}_{1\otimes F^2}(F^3)$ and hence $(\eta_3Q)^{-1}(\eta_2Q)=L(t)L(\epsilon_1s)$ for some unit s in F^2.

Now the same identities that show $\Delta_2\Delta_1=1$ prove (even in the absence of commutativity) that if $P=(\eta_3Q)^{-1}(\eta_2Q)$ then $(\eta_4P)(\eta_2P)(\eta_3P)^{-1}=1$ (cf. the last part of the proof of Lemma 3.7). Thus we have

$$1 = (\epsilon_4t)(\epsilon_2t)(\epsilon_3t)^{-1}\cdot(\epsilon_4\epsilon_1s)(\epsilon_2\epsilon_1s)(\epsilon_3\epsilon_1s)^{-1} = (\Delta_3t)^{-1}(\epsilon_1t)(\epsilon_1\Delta_2s),$$

using (3.6) or, specifically, $\epsilon_{i+1}\epsilon_1=\epsilon_1\epsilon_i$. Since $\Delta_3t=1$ and ϵ_1 is a monomorphism, $t=\Delta_2(s^{-1})$ is a coboundary. This proves that $\{t\}\to\{A(t)\}$ is a monomorphism.

To prove we have an epimorphism, let A' be any algebra split by F, and α any F-algebra isomorphism $A'\otimes F\to\mathrm{End}_F(W)$. By Lemma 3.13(a), we may take $W=V\otimes F$ so that $\alpha(A'\otimes F)=\mathrm{End}_{1\otimes F}(V\otimes F)=E_2$. The image $\alpha(A'\otimes 1)$ is then a subalgebra A of E_2 isomorphic to A' and $\alpha(1\otimes F)=\eta_1L(F)$. We shall eventually produce a cocycle t in F^3 such that the Brauer classes of A and $A(t)$ coincide. Since E_3 decomposes as a tensor product of E_2 and F in two ways, viz.

$$E_3 = (\eta_2E_2)(\eta_3\eta_1L(F)) \cong E_2\otimes F,$$

and

$$E_3 = (\eta_3E_2)(\eta_2\eta_1L(F)) \cong E_2\otimes F,$$

by Lemma 3.2 and Corollary 3.3, and since $E_2\cong A'\otimes F$, it follows that E_3 decomposes into $A'\otimes F\otimes F$ in two ways. Specifically we have a C-algebra isomorphism $\beta_2\colon A'\otimes F\otimes F\to E_3$ defined as the composite of isomorphisms $a'\otimes f\otimes f'\to\alpha(a'\otimes f')\otimes f$ (which is an isomorphism since α is and F is C-flat) and $e\otimes f\to(\eta_2e)(\eta_3\eta_1L(f))$, for $a'\in A'$, f, $f'\in F$, $e\in E_2$. Similarly, β_3 is the composite of isomorphisms $a'\otimes f\otimes f'\to\alpha(a'\otimes f)\otimes f'\in E_2\otimes F$ and $e\otimes f\to(\eta_3e)(\eta_2\eta_1L(f))$. Both β_3 and β_2 send $1\otimes f\otimes f'$ in $A\otimes F\otimes F$ to $\eta_1L(f\otimes f')$ in E_3, so they are in fact F^2-algebra isomorphisms. Hence $\beta_3\beta_2^{-1}$ is an F^2-algebra automorphism of E_3 which, by Lemma 3.13(c), is inner by some P in E_3.

From its definition, this P satisfies $P(\eta_2a)P^{-1}=\eta_3a$ for all a in $A=\alpha(A'\otimes 1)$. That is, $A\subset A(P)$.

We assert $A=A(P)$: by Lemma 3.6, $A(P)\otimes F\cong A(P)\eta_1L(F)$, and, since $A\subset A(P)$, this makes $A(P)\otimes F\cong E_2$, via the isomorphism $a\otimes f\to a\eta_1L(f)$, for $a\in A(P)$, f in F. This same mapping restricted to A is also an isomorphism, by the definition of A. Thus $0=(A(P)\otimes F)/(A\otimes F)\cong(A(P)/A)\otimes F$ as C-modules. But F has a direct summand C, so that $(A(P)/A)\otimes C=0$ and $A(P)=A$.

By Lemma 3.7, $(\eta_4 P)(\eta_2 P)(\eta_3 P)^{-1}$ is a scalar $\eta_1 L(u)$. This u is the desired cocycle associated with A' for, by Lemma 3.10 with $t=1$, $A(P)\otimes A(1)\cong A(u)\otimes E_1$. Since the mapping $\{t\}\to\{A(t)\}$ is a group homomorphism, $A(1)\sim 1$ and so $A(u)\sim A(P)=A\cong A'$, proving Theorem 3.

REMARKS. 1. The proof of Theorem 3 used not so much the fact that C, F and F^2 satisfy (H) but rather the conclusions of Lemma 3.13. Specifically, our mapping of $H^2(F)$ into $\mathfrak{B}(F/C)$ will be a monomorphism if conclusions (b) and (c) ((c) for $R=F$) of Lemma 3.13 are valid. If conclusions (a) and (c) ((c) for $R=F^2$) of Lemma 3.13 hold the mapping is an epimorphism.

2. In view of the last statement of Theorem 2, we have actually proved that, under the hypotheses of Theorem 3, every algebra split by F is equivalent to one (viz. $A(t)$) which contains F as a maximal commutative subalgebra and which can be represented by endomorphisms on $F\otimes F$ commuting with $L(1\otimes F)$ in such a way that the maximal commutative subalgebra becomes $L(F\otimes 1)$.

Now we produce two specific classes of pairs (F, C) for which $H^2(F)\cong\mathfrak{B}(F/C)$ (Theorem 4).

We begin with the following lemma which is a weak generalization of the fact that principal left ideals generated by idempotents are isomorphic modules if and only if they are isomorphic modulo the radical.

LEMMA 3.14. *Let R be a ring, N a radical ideal in R, V_1 and V_2 finitely generated projective R-modules and ρ: $V_1/NV_1\to V_2/NV_2$ an R-isomorphism. Then ρ is induced by an isomorphism $V_1\to V_2$.*

Proof. Let p_i be the natural mapping $V_i\to V_i/NV_i$. Then ρp_1 is an epimorphism of V_1 onto V_2/NV_2. Since V_1 is projective, there exists σ: $V_1\to V_2$ such that $p_2\sigma=\rho p_1$. Since ρ and p_1 are epimorphisms this implies $\sigma V_1+NV_2=V_2$. But N is a radical ideal and $V_2/\sigma V_1$ is finitely generated, so $V_2/\sigma V_1=0$, σ is an epimorphism [7, p. 124, Corollary]. But $\sigma V_1=V_2$ is projective, so σ splits and $\operatorname{Ker}\sigma$ is a direct summand in V_1. This implies $p_1(\operatorname{Ker}\sigma)=\operatorname{Ker}\sigma/N\operatorname{Ker}\sigma$ and also that $\operatorname{Ker}\sigma$ is finitely generated so that $p_1(\operatorname{Ker}\sigma)=0$ only if $\operatorname{Ker}\sigma=0$. But $\rho p_1(\operatorname{Ker}\sigma)=p_2\sigma(\operatorname{Ker}\sigma)=0$ and ρ is an isomorphism, so $\operatorname{Ker}\sigma$ is indeed 0, and σ is an isomorphism.

LEMMA 3.15. *If $C=R[x]$, where R is a commutative ring with minimum condition, or if C is semilocal (i.e., has only finitely many maximal ideals, but is not necessarily Noetherian), then C satisfies hypothesis* (H).

REMARK. If C is semilocal the result was proved by Serre [17, I, Proposition 11] and [18, Proposition 6] assuming C is Noetherian, and by Kaplansky in the general case (unpublished). Serre's proof can easily be modified to take care of the general case. We present such a modification here since [17] seems not to be readily available.

Proof. In the first case, $C=\sum_{\oplus} R_i[x]$, where R_i is a local ring with nilpotent radical. Thus we may assume that R is already local. Let its maximal ideal be N and let M be any finitely generated projective C-module. Then $M/N[x]M$ is a projective module over $R[x]/N[x]=(R/N)[x]$ which is a principal ideal domain. Hence $M/N[x]M$ is free on n generators, say. Let M' be a free C-module on n generators. Then $M/N[x]M\cong M'/N[x]M'$. Since N is nilpotent, so is $N[x]$ and Lemma 3.14 applies to show that $M\cong M'$ and so M is free.

In the second case, since a semilocal ring has only a finite number of maximal ideals, it is a finite direct sum of indecomposable semilocal rings. So we may assume C is indecomposable. We let M be a finitely generated projective C-module and prove that M is free.

For any prime ideal $\mathfrak{p}$ of C, we pass to the local ring $C_{\mathfrak{p}}$. Here $M\otimes C_{\mathfrak{p}}$ is a projective, hence free $C_{\mathfrak{p}}$-module on $n_{\mathfrak{p}}$ generators. If $\mathfrak{p}\supset\mathfrak{p}'$ then $M\otimes C_{\mathfrak{p}'} =(M\otimes_C C_{\mathfrak{p}})\otimes_{C_{\mathfrak{p}}}C_{\mathfrak{p}'}$ so that $n_{\mathfrak{p}}=n_{\mathfrak{p}'}$. Thus we can divide the set of prime ideals of C into disjoint subsets $U_k=\{\mathfrak{p}\mid n_{\mathfrak{p}}=k\}$ such that if $\mathfrak{p}\supset\mathfrak{p}'$, then $\mathfrak{p}\in U_k$ if and only if $\mathfrak{p}'\in U_k$. Of course, k is bounded by the number of generators of M. Let S_k be the complement in C of $\bigcup_{\mathfrak{p}\in U_k}\mathfrak{p}$ and let $\mathfrak{a}_k=\{a\in C\mid as=0 \text{ for some } s \text{ in } S_k\}$.

We assert that the primes containing $\mathfrak{a}_k$ are exactly the primes in U_k. If $\mathfrak{p}\in U_k$ and $a\in\mathfrak{a}_k$ then $as=0$ for some s not in $\mathfrak{p}$, so that $a\in\mathfrak{p}$. Conversely, let $\mathfrak{a}_k\subset\mathfrak{p}_j\in U_j$ and consider the multiplicatively closed set S_kS_j. This does not contain 0; for if we had $s_ks_j=0$ for $s_k\in S_k$, $s_j\in S_j$ we would also have $s_j\in\mathfrak{a}_k\subset\mathfrak{p}_j$, contrary to the definition of S_j. Hence there is a prime ideal $\mathfrak{p}^*$ disjoint from S_kS_j [15, V, Lemma 2]. Then $\mathfrak{p}^*$ is disjoint from both S_k and S_j so that $\mathfrak{p}^*$ is contained in both $\bigcup_{\mathfrak{p}\in U_j}\mathfrak{p}$ and $\bigcup_{\mathfrak{p}\in U_k}\mathfrak{p}$. But $\bigcup_{\mathfrak{p}\in U_j}\mathfrak{p}$ is just the (finite) union of maximal ideals belonging to U_j so that [19, p. 215] $\mathfrak{p}^*\subset\mathfrak{p}$ for some maximal $\mathfrak{p}\in U_j$; thus $\mathfrak{p}^*\in U_j$. Similarly, $\mathfrak{p}^*\in U_k$. This proves $j=k$, proving our assertion.

Now if $j\neq k$, $\mathfrak{a}_j+\mathfrak{a}_k$ cannot be contained in a maximal ideal $\mathfrak{p}\in U_i$ for we have just proved that then $j=i$ and $k=i$. Thus the ideals $\mathfrak{a}_k$ are relatively prime in pairs. Furthermore, if $a\in\bigcap_k\mathfrak{a}_k$ and $\mathfrak{b}$ is the annihilator of a in C, then $\mathfrak{b}$ is an ideal containing at least one element of each S_k. Thus $\mathfrak{b}$ is contained in no prime, which proves $1\in\mathfrak{b}$ and $a=0$. By the Chinese Remainder Theorem, it follows that $C\cong\sum_{\oplus} C/\mathfrak{a}_k$. Since C is indecomposable, there is only one k occurring—the projective module M has the same local rank at every prime. Let M' be a free C-module with this rank. Then $M\otimes C_{\mathfrak{p}}\cong M'\otimes C_{\mathfrak{p}}$ for every $\mathfrak{p}$. If $\mathfrak{p}$ is maximal the residue class field of $C_{\mathfrak{p}}$ is $C/\mathfrak{p}$. We tensor the isomorphism above with $C/\mathfrak{p}$ to get

$$M\otimes C/\mathfrak{p}\cong M'\otimes C/\mathfrak{p}$$

for every maximal $\mathfrak{p}$. If we denote the intersection of the maximal ideals by N then, again by the Chinese Remainder Theorem, we have

$$M/NM \cong M \otimes C/N \cong M \otimes \sum_{\mathfrak{p}\ \max} \oplus\, C/\mathfrak{p} \cong \sum \oplus\, M \otimes C/\mathfrak{p} \cong M'/NM'.$$

By Lemma 3.14, $M \cong M'$ is free.

THEOREM 4. *The mapping* $H^2(F) \to \mathcal{B}(F/C)$ *defined in Theorem 2 is an isomorphism in the following two cases:*

1°. $C = K[x]$, $F = L[x]$, *where* K *is a commutative ring with minimum condition,* L *is a* K*-algebra which is finitely generated and projective as a* K*-module with split unit mapping*(8) $K \to L$ *and where* x *is an indeterminate over* L.

2°. C *is semilocal,* F *is a* C*-algebra which is finitely generated and projective as a* C*-module and such that the unit mapping* $C \to F$ *is a split* C*-module monomorphism*(8).

Proof. In case 1°, L and $L \otimes_K L$ are both finitely generated K-modules, hence satisfy the minimum condition on ideals. Thus C, F and

$$F^2 = L[x] \otimes_{K[x]} L[x] \cong (L \otimes_K L)[x]$$

all fall under the hypotheses of Lemma 3.15. Hence the hypotheses of Theorem 3 are satisfied.

In case 2°, we want to show that F and F^2 are also semilocal so that Lemma 3.15 can be used again to verify the hypotheses of Theorem 3. First, we remark that a commutative ring R is semilocal if and only if R modulo a radical ideal is a ring with minimum condition. Now if C is semilocal with radical N and R (=either F or F^2) is a finitely generated C-module then NR is a radical ideal in R; for if M is any maximal ideal in R then R/M is a finitely generated C-module and $N(R/M) \neq R/M$, by Nakayama's lemma [7, Theorem 1]; therefore $N(R/M) = 0$ and NR is part of the radical of R. Furthermore, R/NR is a finitely generated (C/N)-module and so is a ring with minimum condition. This shows that F and F^2 are semilocal and completes the proof of Theorem 4.

We conclude this section with a discussion of the functorial properties of the homomorphism $H^2(F) \to \mathcal{B}(F/C)$. Both $H^2(F)$ and $\mathcal{B}(F/C)$ are functions of F and C. We assert that they are functors in each of those variables, and that our homomorphism is a mapping of functors.

First, if $C \to D$ is a ring homomorphism, we have a homomorphism $\mathcal{B}(F/C) \to \mathcal{B}(F \otimes D/D)$, defined by associating to each C-algebra A split by F the D-algebra $A \otimes D$ which will be split by $F \otimes D$. We also have a mapping $x \to x \otimes 1$ sending F^n to $F^n \otimes D \cong (F \otimes D) \otimes_D \cdots \otimes_D (F \otimes D) = (F \otimes D)^n$ which gives a mapping of Amitsur complexes. This, in turn, induces a mapping of homology groups $H^n(F) \to H^n(F \otimes D)$, where in the last group $F \otimes D$ is treated as a D-algebra. If we assume that F is finitely generated and projective over C and that $C \to F$ splits, then $F \otimes D$ has the same properties over D and homomorphisms $H^2(F)\to\mathcal{B}(F/C)$ and $H^2(F\otimes D)\to\mathcal{B}(F\otimes D/D)$ may be defined as in Theorem 2. A routine calculation shows that the diagram

$$\begin{array}{ccc} H^2(F) & \rightarrow & H^2(F \otimes D) \\ \downarrow & & \downarrow \\ \mathcal{B}(F/C) & \rightarrow & \mathcal{B}(F \otimes D/D) \end{array}$$

is commutative so that $H^2(F)\to\mathcal{B}(F/C)$ is a mapping of functors.

Our homomorphism is also a mapping of functors of F. If $F'\to F$ is a C-algebra homomorphism then we have an induced homomorphism of the Amitsur complex of F' to the Amitsur complex of F with consequent homomorphisms $H^n(F')\to H^n(F)$. We also have a homomorphism $\mathcal{B}(F'/C)\to\mathcal{B}(F/C)$ by associating to each algebra A split by F' the same algebra; for if A is split by F' so that $A\otimes F'\cong\text{End}_{F'}(W)$, with W a faithful, finitely generated, projective F'-module then A is also split by[12]

$$F\colon A\otimes F\cong(A\otimes F')\otimes_{F'}F\cong\text{End}_F(W\otimes_{F'}F).$$

In fact, $\mathcal{B}(F'/C)\subset\mathcal{B}(F/C)$.

If, besides, F' and F satisfy the hypotheses of Theorem 2 we have mappings $H^2(F')\to\mathcal{B}(F'/C)$ and $H^2(F)\to\mathcal{B}(F/C)$. We sketch a proof of the commutativity of the diagram

$$\begin{array}{ccc} H^2(F') & \rightarrow & H^2(F) \\ \downarrow & & \downarrow \\ \mathcal{B}(F'/C) & \rightarrow & \mathcal{B}(F/C) \end{array}$$

so that the homomorphism $H^2(F)\to\mathcal{B}(F/C)$ is again a mapping of functors. The cocycle t' in F'^3 maps to a cocycle t in F^3, and we must prove that $A(t')\sim A(t)$. Now the mapping $F'\to F$ induces mappings $F'^3\to F'\otimes F^2$ and hence $\text{End}_{1\otimes F'^2}(F'^3)\to\text{End}_{1\otimes F^2}(F'\otimes F^2)$. Let the image of the endomorphism $L(t')$ be P. Then the image of $A(t')$ under $A(t')\to A(t')\otimes F\cong A(t')\otimes F'\otimes_{F'}F\cong\text{End}_{1\otimes F}(F'\otimes F)$ is exactly $A(P)$ as defined in (3.7) with $V=F'$. Finally, $(\eta_4P)(\eta_2P)(\eta_3P)^{-1}$ turns out to be $\eta_1L(t)$ and so Lemma 3.10, with u replaced by the present t and t by 1, shows $A(t')\cong A(P)\sim A(t)$.

We claim that both $H^n(F)$ and $\mathcal{B}(F/C)$ are functors of F that commute with certain direct limits: if $\{F_\alpha,\phi_{\alpha\beta}\}$ is a direct system of C-algebras and homomorphisms with limit F then the Amitsur complex of F is the limit of the Amitsur complexes of the F_α, because $\otimes$ commutes with direct limits [9, VI, Exercise 17]. Since the homology functor also commutes with direct limits [9, V, Proposition 9.3*], we have $H^n(F)=\lim_{\rightarrow} H^n(F_\alpha)$. The analogous argument for $\mathcal{B}(F/C)$ is slightly more complex and will be needed in §5; hence we state it as

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(12) It is clear that $W\otimes_{F'}F$ is a finitely generated projective F-module. That it is faithful follows from [6, Theorems A.3 and A.5].

LEMMA 3.16. *Let F be the union of subalgebras F_α such that each F_α is a direct summand in F as F_α-module. Then $\mathfrak{B}(F/C)$ is the union of the groups $\mathfrak{B}(F_\alpha/C)$.*

Proof. As above, $\mathfrak{B}(F_\alpha/C)\subset\mathfrak{B}(F/C)$. It remains to show that if an algebra A is split by F, then it is split by some F_β.

Let $A\otimes F\cong E=\operatorname{End}_F(W)$, where W is a faithful projective F-module generated by $w_1, \cdots, w_n$. Let $u_1, \cdots, u_n$ be free generators of a free F-module containing W as a direct summand. Since the expression of the w's in terms of the u's involves only finitely many coefficients in F, these coefficients all lie in some F_α. Then $F_\alpha u_1+\cdots+F_\alpha u_n$ is a free F_α-module containing $W_\alpha=F_\alpha w_1+\cdots+F_\alpha w_n$ as a direct summand. It follows that $W\cong W_\alpha\otimes_{F_\alpha}F$, so that $E\cong\operatorname{End}_{F_\alpha}(W_\alpha)\otimes_{F_\alpha}F$. We let E_α denote the subset of E corresponding to $\operatorname{End}_{F_\alpha}(W_\alpha)$ under this isomorphism so that $E_\alpha F=E$. Then E_α is a finitely generated F_α-module and an argument like the construction of W_α above shows that, under the isomorphism $E\rightarrow A\otimes F$, the image of E_α is contained in some $A\otimes F_\beta$ (without loss of generality, we take $\beta>\alpha$). Hence $E_\beta=E_\alpha\cdot F_\beta$ also goes into an F_β-submodule E_β' of $A\otimes F_\beta$. If $B=(A\otimes F_\beta)/E_\beta'$ then $B\otimes_{F_\beta}F=(A\otimes F_\beta\otimes_{F_\beta}F)/(E_\beta'\otimes_{F_\beta}F)=A\otimes F/E_\beta'F=0$, since $E_\beta F=E$ and $E_\beta'F=\operatorname{Im}(E\rightarrow A\otimes F)=A\otimes F$. Since $F_\beta\rightarrow F$ splits, we have also $B=0$ and $A\otimes F_\beta=E_\beta'\cong E_\beta\cong\operatorname{End}_{F_\beta}(W_\beta)$, with $W_\beta=W_\alpha\otimes_{F_\alpha}F_\beta$. Thus A is split by F_β, proving the lemma.

COROLLARY 3.17. *Let F be a C-algebra which is the union of subalgebras F_α and assume that the F_α-module monomorphism $F_\alpha\rightarrow F$ splits for all α. Assume further that the pairs (F_α, C) satisfy the hypothesis of Theorem 2. If the mappings $H^2(F_\alpha)\rightarrow\mathfrak{B}(F_\alpha/C)$ are isomorphisms for each α, then $H^2(F)\cong\mathfrak{B}(F/C)$.*

Proof. Since our homomorphism $H^2(F_\alpha)\rightarrow\mathfrak{B}(F_\alpha/C)$ is a mapping of functors and both functors commute with this kind of direct limit, we have $H^2(F)=\lim_{\rightarrow} H^2(F_\alpha)\cong\lim_{\rightarrow}\mathfrak{B}(F_\alpha/C)\cong\mathfrak{B}(F/C)$.

4. **The purely inseparable case.** If F is a purely inseparable extension field of the field C and has exponent one and finite degree over C, Hochschild [13] exhibited an isomorphism of $\mathfrak{B}(F/C)$ with the group $\mathcal{E}(T, F)$ of regular[13] restricted Lie algebra extensions of F by T, where T is the Lie algebra of C-derivations of F. In this section, we shall produce an isomorphism $H^2(F)\rightarrow\mathcal{E}(T, F)$, which, combined with Theorem 3, will give another proof of Hochschild's result.

In [3, §7], Amitsur attempted this unsuccessfully. For one thing, he

(13) "Regular" here means that any such extension S has a structure as left F-module compatible with the Lie multiplication and pth-power operation [11, pp. 481–482]. We do not need to make this more precise; it is sufficient to note that if S is a restricted special Lie subalgebra of an associative algebra A (i.e. if s_1, s_2 are in S, $[s_1, s_2]=s_1\cdot s_2-s_2\cdot s_1$ and $s^{[p]}=s^p$ where the dot and pth power on the right sides of the equations are the associative products in A), and if A contains a subalgebra F with $FS\subset S$, then S will be regular.

tried to establish an isomorphism of $H^n(F)$ with a subgroup of the restricted cohomology group $H^n_*(T, F) = \text{Ext}_{U^*}^n(C, F)$ [12], where U^* is the restricted universal enveloping algebra of the Lie algebra T. Now $H^2_*(T, F)$ can be zero when $\mathcal{B}(F/C)$ is not; for example, when C is the field of rational functions in two indeterminates x, y over $GF(2)$ and $F = C(x^{1/2})$. (In this case F is U^*-injective, so $H^n_*(T, F) = 0$; one way of showing that $\mathcal{B}(F/C) \neq 0$ for these F and C is by using the proof of Theorem 6.) Secondly, [3, Theorem 7.2] asserts erroneously that $H^n(F^+) \cong H^n(F)$, where $H^n(F^+)$ is the nth cohomology group of the complex $C \to F \to F^2 \to \cdots$ with coboundary operator

$$\Delta_n^+ : F^n \to F^{n+1}$$

defined by $\Delta_n^+ = \sum_{i=1}^{n+1} (-1)^{i-1}\epsilon_i$ for $n \geqq 0$. As a matter of fact,

LEMMA 4.1. *Let C be a commutative ring and F a C-algebra such that the unit mapping $C \to F$ splits*(8). *Then $H^n(F^+) = 0$ for $n \geqq 0$. If, besides, F is C-flat, then*

for $q \in F^2$, $(\epsilon_2 - \epsilon_3)q = 0$ if and only if $q \in F \otimes 1$;

for $r \in F^3$, $(\epsilon_2 - \epsilon_3 + \epsilon_4)r = 0$ if and only if $r = (\epsilon_2 - \epsilon_3)q$ for some $q \in F^2$.

Proof. That $\Delta_{n+1}^+ \Delta_n^+ = 0$ is proved as usual, using (3.3) (cf. [3, Theorem 5.1]), so that we do indeed have a complex. Let $\phi: F \to C$ be the mapping postulated in the lemma and define homomorphisms $s_n: F^n \to F^{n-1}$ by $s_n(x_1 \otimes \cdots \otimes x_n) = \phi(x_1)(x_2 \otimes \cdots \otimes x_n)$ for $n > 0$, and $s_0 = 0$. Then $s_{n+1}\Delta_n^+ + \Delta_{n-1}^+ s_n = \text{identity}$ (s_n defines a contracting homotopy) which implies $H^n(F^+) = 0$ for $n \geqq 0$. If F is C-flat, then the complex $F \otimes C \to F \otimes F \to F \otimes F^2 \to \cdots$ with coboundary operators $1 \otimes \Delta_n^+ : F \otimes F^n \to F \otimes F^{n+1}$ will also have zero cohomology groups. But $1 \otimes \Delta_n^+ = \epsilon_2 - \epsilon_3 + \epsilon_4 - \cdots$, which proves the rest of the lemma.

Now let C be a ring of prime characteristic p, F any C-algebra, and T the set of all derivations of F over C. Just as in the case where C is a field, T is a Lie algebra with a pth-power operation and an F-module structure. For reasons which will become apparent shortly, we define, for any D in T, a derivation ${}_nD$ of the algebra F^n over $1 \otimes F^{n-1}$ by(14)

$$(4.1) \qquad {}_nD(x_1 \otimes \cdots x_n) = Dx_1 \otimes x_2 \cdots \otimes x_n.$$

It is clear that, if $i > 1$ and $x \in F^n$, then

$$(4.2) \qquad {}_{n+1}D(\epsilon_i x) = \epsilon_i({}_nD(x)), \qquad {}_{n+1}D(\epsilon_1 x) = 0.$$

Although our results are logically independent of those of [13], the latter immediately suggest an explicit isomorphism between $H^2(F)$ and $\mathcal{E}(T, F)$ in case F is a purely inseparable field of exponent one over C. If A is a central

(14) If F is a purely inseparable extension field of exponent one, [10, Proposition 2.3] shows that any way of extending the action of T to F^n is equivalent to (4.1).

simple C-algebra containing a maximal subfield G isomorphic to F, Hochschild associates to A a special Lie algebra S which is an extension of F by T as follows: $S=\{a\in A \mid ag-ga\in G \text{ for all } g\in G\}$; $S\to T$ by associating to a in S the derivation $g\to ag-ga=[a, g]$ of G (or rather the derivation of F which corresponds to this by the isomorphism of G and F); the kernel of the mapping $S\to T$ is clearly G, which is isomorphic to F.

If we apply the construction to $A=A(t)$ and $G=L(F\otimes 1)$, we get $S=\{a\in A(t) \mid [a, L(f\otimes 1)]\in L(F\otimes 1) \text{ for all } f\in F\}$, and for every $a\in S$ we get a derivation $D\in T$ such that

$$(4.3)\qquad [a, L(f\otimes 1)] = L(Df\otimes 1) = L({}_2D(f\otimes 1)).$$

Now, for any derivation δ of a ring and any ring element x,

$$[\delta, L(x)] = L(\delta x)$$

so that (4.3) becomes $[a, L(f\otimes 1)]=[{}_2D, L(f\otimes 1)]$ and $a-{}_2D$ is an endomorphism of F^2 commuting with $L(F\otimes 1)$. By (4.1), $[{}_2D, L(1\otimes F)]=L({}_2D(1\otimes F))=0$, and $[a, L(1\otimes F)]=0$, because $a\in \operatorname{End}_{1\otimes F}(F^2)$. Hence $a-{}_2D$ commutes with $L(F^2)$. Thus there is an element q in F^2 with $a={}_2D+L(q)$. If we substitute this formula for a into the defining equation for $A(t)$:

$$(4.4)\qquad L(t)(\eta_2 a) = (\eta_3 a)L(t)$$

and use $\eta_i({}_2D)={}_3D$, $i=2, 3$, we obtain $[{}_3D, L(t)]=L(t)(\eta_2L(q)-\eta_3L(q))$, or, by (3.4), $L({}_3Dt)=L(t)L(\epsilon_2q-\epsilon_3q)$, which finally becomes

$$(4.5)\qquad {}_3Dt/t = (\epsilon_2-\epsilon_3)q.$$

Conversely, if $D\in T$ and q is any element of F^2 satisfying (4.5) then the element $a={}_2D+L(q)$ in $\operatorname{End}_{1\otimes F}(F^2)$ satisfies (4.4) and so lies in $A(t)$. Thus the Lie algebra S is simply the set of all endomorphisms of F^2 of the form ${}_2D+L(q)$, with D ranging over T and q satisfying (4.5). Hence we make the

Definition. For every cocycle t in F^{3*}, let $\Theta(t)$ be the set of endomorphisms of F^2 of the form ${}_2D+L(q)$, with D ranging through T and q an element of F^2 satisfying ${}_3Dt/t=(\epsilon_2-\epsilon_3)q$.

Theorem 5. *Let C be a ring of prime characteristic p, F a C-algebra which is a flat C-module with a split unit mapping*[8], *and T the restricted Lie algebra of derivations of F over C. Then the mapping Θ induces a homomorphism of $H^2(F)$ into $\mathcal{E}(T, F)$, the group of regular restricted Lie algebra extensions of F by T.*

The proof will consist of a series of lemmas.

Lemma 4.2. *Let C and F satisfy the hypotheses of Theorem 5 (except that we need not assume characteristic p) and let D be an element of T. Define a map α_D of the graded group $\sum_{\oplus} F^{n*}$ into the graded group $\sum_{\oplus} F^n$ by*

$$\alpha_D(x) = {}_nDx/x, \qquad \text{for } x \text{ in } F^{n*}.$$

Then α_D is a homomorphism of graded groups and, for $x \in F^{n*}$, $\alpha_D(\Delta_n x) = -(1 \otimes \Delta_{n-1}^{+})\alpha_D(x)$ (with $1 \otimes \Delta_{n-1}^{+}$ as in the proof of Lemma 4.1). Therefore α_D is a homomorphism of the Amitsur complex into the complex $\{F^n, -1 \otimes \Delta_{n-1}^{+}\}$. In particular, there exists an element q in F^2 satisfying (4.5). Another element q' in F^2 satisfies (4.5) with the same D if and only if $q' - q \in F \otimes 1$.

Proof. Since α_D is a "logarithmic derivative" a standard computation shows that, for $x, y \in F^{n*}$, $\alpha_D(xy) = \alpha_D(x) + \alpha_D(y)$ and $\alpha_D(\Delta_n x) = -\epsilon_2({}_nDx/x) + \epsilon_3({}_nDx/x) + \cdots$, proving the first part. If t is a cocycle in F^{3*} we conclude $\epsilon_2\alpha_D(t) - \epsilon_3\alpha_D(t) + \epsilon_4\alpha_D(t) = 0$. By Lemma 4.1, this is equivalent to $\alpha_D(t) = (\epsilon_2 - \epsilon_3)q$ for some q in F^2. Another q' in F^2 satisfies the same relation if and only if $(\epsilon_2 - \epsilon_3)(q - q') = 0$ which, again by Lemma 4.1, is equivalent to $q - q' \in F \otimes 1$.

LEMMA 4.3. *Let D and D' be derivations of a commutative ring R of prime characteristic p, let q and q' be elements of R and let $L(q)$ denote left multiplication by q on R. Then*

$$[D + L(q), D' + L(q')] = [D, D'] + L(Dq' - D'q);$$
$$(D + L(q))^p = D^p + L(q^p + D^{p-1}q).$$

Proof. The first equality follows directly from definitions; the second may be found in [14, p. 223], [12, p. 560] or [10, p. 201].

LEMMA 4.4. *Let ω be a C-module homomorphism of T into F^n and define, for every D in T, $r(D) = {}_nD + L(\omega(D))$ so that r is a module homomorphism of T into $E_n = \mathrm{End}_{1\otimes F^{n-1}}(F^n)$. Then r is a Lie algebra homomorphism if and only if*

$$\omega([D, D']) = {}_nD(\omega(D')) - {}_nD'(\omega(D)) \text{ for all } D, D' \text{ in } T; \tag{4.6}$$

r preserves pth powers if and only if

$$\omega(D^p) = \omega(D)^p + {}_nD^{p-1}(\omega(D)) \text{ for all } D \text{ in } T. \tag{4.7}$$

These conditions are satisfied if $\omega(D) = {}_nDt/t$ for some fixed unit t in F^n.

Proof. These statements appear in [10, Chapter 2, (33), (35), (41), (42′) and the necessity part of Proposition 2.7]. Cartier's proofs are given in the case where $n = 1$ and F is a purely inseparable extension field of exponent 1. However, the proofs involve only formal identities and are equally valid in our case. The first two statements in Lemma 4.4 are immediate corollaries of Lemma 4.3.

LEMMA 4.5. *$\Theta(t)$ is a restricted special Lie subalgebra of* $\mathrm{End}_{1\otimes F}(F^2)$.

Proof. Using Lemma 4.3, we see that we are required to show

$$(\epsilon_2 - \epsilon_3)({}_2Dq' - {}_2D'q) = [{}_3D, {}_3D']t/t,$$

and

$$(\epsilon_2 - \epsilon_3)(q^p + {}_2D^{p-1}q) = {}_3D^pt/t,$$

whenever $(\epsilon_2-\epsilon_3)q={}_3Dt/t$ and $(\epsilon_2-\epsilon_3)q'={}_3D't/t$. This is immediate from (4.2) and the last part of Lemma 4.4.

LEMMA 4.6. *$\Theta(t)$ is a regular restricted extension of F by T.*

Proof. $\Theta(t)$ is a left F-module since $L(f\otimes 1)({}_2D+L(q))={}_2(fD)+L((f\otimes 1)q)$. Since, by Lemma 4.5, $\Theta(t)$ is a special restricted Lie subalgebra of $\mathrm{End}_{1\otimes F}(F^2)$, it follows by Footnote 13 that $\Theta(t)$ is regular. The mapping ${}_2D+L(q)\rightarrow D$ of $\Theta(t)$ to T is an epimorphism of regular, restricted Lie algebras, by Lemma 4.3. The kernel is the set of $L(q)$ with q satisfying (4.5) for $D=0$. By Lemma 4.1, this is $L(F\otimes 1)\cong F$.

We recall the definition of the sum S_1+S_2 of two regular extensions S_1, S_2 of F by T as given in [12]: let E be the Lie subalgebra of $S_1\oplus S_2$ consisting of the pairs (s_1, s_2) with s_1 and s_2 mapping to the same element of T. Let J be the ideal of E consisting of the elements of the form $(f, -f)$ with $f\in F\subset S$. Then S_1+S_2 is defined to be E/J, again a regular, restricted extension of F by T with the copy of F in E/J being the image of $F\oplus F$ in E/J.

LEMMA 4.7. *If t_1, t_2 are cocycles in F^{3*} then $\Theta(t_1t_2)$ is isomorphic to $\Theta(t_1)+\Theta(t_2)$ as a regular restricted Lie algebra.*

Proof. In $\Theta(t_1)\oplus\Theta(t_2)$, we consider the subalgebra E of all pairs

$$({}_2D + L(q_1), {}_2D + L(q_2)) \quad \text{with} \quad {}_3Dt_i/t_i = (\epsilon_2 - \epsilon_3)q_i, \qquad i = 1, 2.$$

Let ρ be defined by

$$\rho({}_2D + L(q_1), {}_2D + L(q_2)) = {}_2D + L(q_1 + q_2).$$

Since $(\epsilon_2-\epsilon_3)(q_1+q_2)={}_3D(t_1t_2)/t_1t_2$, $\rho(E)$ is a Lie subalgebra of $\Theta(t_1t_2)$. Now suppose that ${}_2D+L(q)$ is in $\Theta(t_1t_2)$, so that $(\epsilon_2-\epsilon_3)q={}_3Dt_1/t_1+{}_3Dt_2/t_2$. By Lemma 4.2, there are elements q_i in F^2 with $(\epsilon_2-\epsilon_3)q_i={}_3Dt_i/t_i$, $i=1, 2$. Thus, by Lemma 4.1, $-q+(q_1+q_2)=f\otimes 1$ for some f in F. Hence $\rho({}_2D+L(q_1-f\otimes 1), {}_2D+L(q_2))={}_2D+L(q)$ and $\rho(E)=\Theta(t_1t_2)$. Using our earlier formulas for the operation of F, the Lie product, and pth-power operation in $\Theta(t_1)$, $\Theta(t_2)$ and $\Theta(t_1t_2)$, it is easily verified that ρ is an F-linear, restricted Lie algebra homomorphism of E onto $\Theta(t_1t_2)$.

Finally, if $\rho({}_2D+L(q_1), {}_2D+L(q_2))=0$, we have ${}_2D+L(q_1+q_2)=0$. Applying the left side to $1\otimes 1$ shows that $q_1=-q_2$ and so $D=0$. Thus $(\epsilon_2-\epsilon_3)q_i=0$, $i=1, 2$, and, by Lemma 4.1, q_i is in $F\otimes 1$. Hence $\mathrm{Ker}\ \rho=\{(L(f\otimes 1), L(-f\otimes 1))\}=J$, proving Lemma 4.7.

LEMMA 4.8. *If t is a coboundary in F^{3*} then $\Theta(t)$ is a split, restricted, regular extension of F by T.*

Proof. There is an element s in F^{2*} with $t=\Delta_2(s^{-1})$. By Lemma 4.2,

$${}_3Dt/t = (\epsilon_2 - \epsilon_3){}_2Ds/s, \qquad \text{for all } D \text{ in } T.$$

By Lemma 4.2 this means $\Theta(t)=\{{}_2D+L({}_2Ds/s+f\otimes 1)\}$, where D runs through T and f through F. By Lemma 4.4, $D\to{}_2D+L({}_2Ds/s)$ is an F-linear, restricted, Lie algebra monomorphism of T into $\Theta(t)$, so that $\Theta(t)$ is split.

This completes the proof of Theorem 5.

THEOREM 6. *Let C be a field and F a purely inseparable extension field of exponent* 1 *and finite degree over C. Then the homomorphism $H^2(F)\to\mathcal{E}(T, F)$ in Theorem* 5 *is an isomorphism.*

Proof. Given any regular extension S of F by T, there is an F-linear Lie algebra isomorphism ϕ of T into S which is an inverse of the given projection $S\to T$. That is, $S=\phi(T)+F$ (F-module direct sum). Moreover, $g(D)=(\phi(D))^p-\phi(D^p)$ is an additive function from T to C such that $g(fD)=f^pg(D)$ for every f in F [13, Theorem 4]. Thus S is isomorphic to the set of all ordered pairs (D,f) with D in T and f in F. The operations are given by

$$\begin{aligned}[][(D,f),(D',f')] &= ([D, D'], Df' - D'f),\\ (D,f)^p &= (D^p, f^p + D^{p-1}f + g(D)),\\ f'(D,f) &= (f'D, f'f), \qquad\qquad [13, \text{pp. } 487\text{–}488].\end{aligned}$$

Given S, ϕ and hence g, we shall construct a cocycle t such that $\Theta(t)$ is equivalent to S. To do this, we recall the structure of a purely inseparable extension field of exponent one: $F=C[a_1, \cdots, a_k]$ with a_i^p in C, and T has a left F-basis given by the derivations $D_1, \cdots, D_k$ determined by $D_i(a_j)=\delta_{ij}$. These D_i's have the additional property $D_i^p=0$ [10, Chapter 2, §2]. If we consider F^2 as an F-module by virtue of multiplication by $F\otimes 1$, we can define an F-linear function λ from T to F^2 by

$$\lambda(D_i) = g(D_i)\overset{+}{\Delta}_1(a_i^{p-1}) \tag{4.8}$$

where $\overset{+}{\Delta}_1=\epsilon_1-\epsilon_2$ is the coboundary operator on F introduced above Lemma 4.1. We proceed to verify that $\nu\colon S\to\mathrm{End}_{1\otimes F}(F^2)$, defined by

$$\nu(D,f) = {}_2D + L(\lambda(D) + f \otimes 1),$$

is an equivalence of regular, restricted Lie algebra extensions of F by T which carries the splitting map ϕ into the map $D\to{}_2D+L(\lambda(D))$. Furthermore, the image of ν is $\Theta(t)$, where t is defined by([15])

$$\begin{aligned} t &= \prod_{i=1}^{k} \exp s_iu_i,\\ s_i &= \epsilon_1\lambda(D_i) = \epsilon_1\overset{+}{\Delta}_1(g(D_i)a_i^{p-1}) = g(D_i)(1 \otimes 1 \otimes a_i^{p-1} - 1 \otimes a_i^{p-1} \otimes 1),\\ u_i &= -\epsilon_3\overset{+}{\Delta}_1(a_i) = a_i \otimes 1 \otimes 1 - 1 \otimes a_i \otimes 1.\end{aligned} \tag{4.9}$$

([15]) As usual, if R is an algebra over $GF(p)$ and r in R satisfies $r^p=0$, then we define $\exp r=1+r/1!+\cdots+r^{p-1}/(p-1)!$.

By Lemma 4.3 and a standard calculation, the condition that ν preserve the Lie product is

$$\lambda([D, D']) = {}_2D(\lambda(D')) - {}_2D'(\lambda(D)). \tag{4.10}$$

Since $[D_i, D_j]=0$ and ${}_2D_j\lambda(D_i)=0$ for $i\neq j$, (4.10) is satisfied when D and D' are basis elements of T. Since both sides of (4.10) are bilinear in D and D', (4.10) is true for all D, D' in T. Note that (4.10) is Cartier's condition $d\lambda=0$ [10, p. 201, (34)].

The condition that ν preserve pth powers is

$$[{}_2D + L(\lambda(D) + f \otimes 1)]^p = {}_2D^p + L(\lambda(D^p) + f^p \otimes 1 + D^{p-1}f \otimes 1 + g(D) \otimes 1).$$

Using Lemma 4.3, this translates to

$$\lambda(D)^p + {}_2D^{p-1}(\lambda(D)) - \lambda(D^p) = g(D) \otimes 1. \tag{4.11}$$

If $D=D_i$, then $\lambda(D_i)^p=0=\lambda(D_i^p)$ and ${}_2D_i^{p-1}(\lambda(D_i))=-g(D_i)(D_i^{p-1}a_i^{p-1}\otimes 1)$ $=-g(D_i)((p-1)!\otimes 1)=g(D_i)\otimes 1$, by Wilson's theorem, which verifies (4.11) for $D=D_i$. Once again, both sides of (4.11) are p-semilinear([16]) (the left hand side in Cartier's notation is $\lambda(D)^p-(C\lambda(D))^p$; since we showed above that $d\lambda=0$, we can use Cartier's computations [10, p. 203, (44) and (45)] to insure the p-semilinearity of this function). Thus (4.11) is true for all D in T.

Clearly, ν is F-linear, has kernel zero, is an equivalence of restricted, regular extensions and carries $\phi: D\rightarrow(D, 0)$ into the mapping $D\rightarrow\nu(D, 0)$ $={}_2D+L(\lambda(D))$. It remains only to compute the image of ν. By the definition of Θ, we are required to prove that t is a cocycle in F^{3*} (it is evidently an element of F^{3*}) and that, for every $D\in T$,

$$ {}_3Dt/t = (\epsilon_2 - \epsilon_3)\lambda(D), \qquad \text{i.e., } \alpha_D(t) = (1 \otimes \overset{+}{\Delta}_1)(\lambda(D)). \tag{4.12}$$

Again, it is sufficient to verify (4.12) for $D=D_j$.

With notations as in (4.9), we have ${}_3Ds_i=0$ for all i and all D by (4.2). Furthermore, ${}_3D_j(u_i)$ is 0 for $i\neq j$ and $1\otimes 1\otimes 1$ for $i=j$. Thus ${}_3D_j(\exp s_iu_i)$ is 0 for $i\neq j$; for $i=j$, because $s_j^p=0$, we can differentiate as for ordinary power series to get ${}_3D_j(\exp s_ju_j)=s_j \exp s_ju_j$. Hence $\alpha_{D_j}(t)=\sum_i \alpha_{D_j}(\exp s_iu_i)=s_j$ $=\epsilon_1\lambda(D_j)$. But $\lambda(D_j)\in\Delta_1^+(F)$ so that $\Delta_2^+(\lambda(D_j))=0$ and $s_j=\epsilon_1\lambda(D_j)$ $=(\epsilon_2-\epsilon_3)\lambda(D_j)$, as desired.

To prove $\Delta_3 t=1$, it will suffice to show $\Delta_3(\exp s_ju_j)=1$ for all j. Therefore, we drop the subscripts j. Because exp is defined by a truncated power series, the usual multiplicative property does not hold without restriction. However, it is easy to see that $\exp(a+b)=(\exp a)(\exp b)$ if a and b in F^3 are both multiples of a single nilpotent element of index p. We shall need this remark, together with the obvious fact that $\epsilon_i \exp=\exp \epsilon_i$, to compute

([16]) If $h: T\rightarrow F^2$, we say h is p-semilinear if $h(D+D')=h(D)+h(D')$ and $h(fD)=(f\otimes 1)^p h(D)$ for f in F and D, D' in T.

$$\Delta_3(\exp su) = \prod_{i=1}^{4} (\exp(\epsilon_i s)(\epsilon_i u))^{(-1)^{i-1}}.$$

Since s is in $\epsilon_1 F^2$, $\epsilon_1 s = \epsilon_2 s$, by (3.3). Furthermore, from its definition (4.9), $s^p = 0$. Thus

$$\exp((\epsilon_1 s)(\epsilon_1 u)) \exp((\epsilon_2 s)(\epsilon_2 u))^{-1} = \exp((\epsilon_2 s)(\epsilon_1 u - \epsilon_2 u)).$$

Similarly, $\epsilon_3 u = \epsilon_4 u$ and $u^p = 0$, so

$$\exp((\epsilon_3 s)(\epsilon_3 u)) \exp((\epsilon_4 s)(\epsilon_4 u))^{-1} = \exp((\epsilon_3 s - \epsilon_4 s)(\epsilon_3 u)).$$

Furthermore, $s = \epsilon_1 \Delta_1^+(a)$ with $a \in F$, so, by (3.3), $\epsilon_i s = \epsilon_1 \epsilon_{i-1} \Delta_1^+(a)$ for $i > 1$, and $\epsilon_2 s - \epsilon_3 s + \epsilon_4 s = \epsilon_1 \Delta_2^+ \Delta_1^+(a) = 0$. Similarly, since $u = \epsilon_3 \Delta_1^+(b)$, we have $\epsilon_1 u - \epsilon_2 u + \epsilon_3 u = \epsilon_4 \Delta_2^+ \Delta_1^+(b) = 0$. It follows that $(\epsilon_2 s)(\epsilon_1 u - \epsilon_2 u) = (\epsilon_3 s - \epsilon_4 s)(-\epsilon_3 u)$, so that the exponentials on the right sides of the equations above are reciprocals of each other and $\Delta_3(\exp su) = 1$.

To complete the proof that Θ gives a one-to-one correspondence, it will clearly be sufficient to show that if S is already $\Theta(t_1)$ for some t_1 in F^3, then t is cohomologous to t_1. If $S = \Theta(t_1)$, the extension map $S \to T$ is ${}_2D + L(q) \to D$ and any inverse map ϕ: $T \to S$ must be of the form

$$\phi(D) = {}_2D + L(\mu(D)),$$

with $\mu \in \mathrm{Hom}_F(T, F^2)$ if ϕ is to be F-linear. If ϕ is to be a Lie algebra homomorphism besides, then μ must satisfy the condition on ω in (4.6). Now the equivalence ν: $S \to \Theta(t)$ carries $\phi(D) = {}_2D + L(\mu(D))$ to ${}_2D + L(\lambda(D))$ and preserves the Lie product and pth powers. Therefore λ and also $\omega = \mu - \lambda$ satisfy (4.6); and $({}_2D + L(\mu(D)))^p - ({}_2D^p + L(\mu(D^p))) = ({}_2D + L(\lambda(D)))^p - ({}_2D^p + L(\lambda(D^p))) = g(D) \otimes 1$. Lemma 4.3 then implies

$$\mu(D)^p + {}_2D^{p-1}(\mu(D)) - \mu(D^p) = \lambda(D)^p + {}_2D^{p-1}(\lambda(D)) - \lambda(D^p),$$

so that $\omega = \mu - \lambda$ satisfies (4.7). Now we imitate the proof of [10, Proposition 2.7] to show that ω is a logarithmic derivative: define $r(D) = {}_2D + \omega(D)$. By Lemma 4.4, r is a restricted Lie algebra homomorphism of T to the endomorphisms of F^2, and thus verifies the hypotheses of [10, Proposition 2.3], where F^2 is considered an F-space by virtue of the action of $F \otimes 1$. If U denotes the subset of F^2 annihilated by $r(T)$ then [10, Proposition 2.3] asserts that $(F \otimes 1)U = F^2$. If U contained no unit, U would be contained in the radical of the local algebra F^2, contradicting $(F \otimes 1)U = F^2$. Hence U contains a unit u so that ${}_2Du + \omega(D)u = 0$ for every D in T. If $w = u^{-1}$, then $\omega(D) = {}_2Dw/w$ for every D in T. Next, by (4.12),

$$ {}_3Dt/t = \alpha_D(t) = (1 \otimes \Delta_1^+)\lambda(D)$$

and similarly

$$\alpha_D(t_1) = (1 \otimes \overset{+}{\Delta_1})\mu(D).$$

Hence

$$\alpha_D(tt_1^{-1}) = \alpha_D(t) - \alpha_D(t_1) = (1 \otimes \Delta_1^+)\omega(D) = (1 \otimes \Delta_1^+)(\alpha_D w) = -\alpha_D(\Delta_2 w)$$

by Lemma 4.2. Thus $\alpha_D(tt_1^{-1}\Delta_2 w) = 0$ and ${}_3D(tt_1^{-1}\Delta_2 w) = 0$ for all D in T. This means $tt_1^{-1}\Delta_2 w \in 1 \otimes F^2$. Write $tt_1^{-1}\Delta_2 w = \sum 1 \otimes x_i \otimes y_i$ with x_i and y_i in F. Since the product of cocycles is a cocycle, $\Delta_3(\sum 1 \otimes x_i \otimes y_i) = 1$ which gives

$$\sum 1 \otimes x_i \otimes 1 \otimes y_i = \sum 1 \otimes x_i \otimes y_i \otimes 1.$$

Applying the contraction $f_1 \otimes f_2 \otimes f_3 \otimes f_4 \to f_1 \otimes f_2 f_3 \otimes f_4$ of F^4 into F^3, we get $\sum 1 \otimes x_i \otimes y_i = \sum 1 \otimes x_i y_i \otimes 1$ so that

$$tt_1^{-1}\Delta_2 w = 1 \otimes f \otimes 1 = \Delta_2(1 \otimes f)$$

with $f = \sum x_i y_i$. Hence

$$t = t_1\Delta_2((1 \otimes f)/w).$$

This completes the proof of Theorem 6.

COROLLARY. *Let A be a central simple C-algebra split by $F = C[a_1, a_2, \cdots, a_k]$, a_i^p in C. Then A is similar to $A_1 \otimes A_2 \otimes \cdots \otimes A_k$ where A_i is a central simple C-algebra split by $C[a_i]$* (cf. [2, *Chapter VII, Theorems* 28 *and* 25]).

Proof. The class of A corresponds to a cohomology class $\{t\}$. By the proof of Theorem 6, this class contains a cocycle t of the form (4.9). Since $t = \prod t_i$ with t_i a cocycle of $(C[a_i])^{3*}$, the desired result follows from Theorem 3.

5. **Polynomial rings.** The techniques of §§3, 4 can be used to prove a theorem (Theorem 8) of Auslander-Goldman on the relation between the full Brauer groups $\mathcal{B}(K[x])$ and $\mathcal{B}(K)$ when K is a field. We consider the mapping $K[x] \to K$ defined by $f(x) \to f(0)$. According to the last part of §3, we then have several induced homomorphisms for every field extension L of finite degree over K. First, α: $\mathcal{B}(L[x]/K[x]) \to \mathcal{B}(L/K)$ [17], which clearly can be extended to a homomorphism of the full Brauer groups, α: $\mathcal{B}(K[x]) \to \mathcal{B}(K)$. Next, we have algebra homomorphisms α': $L[x]^n \to L^n$ which in this case may be made explicit by identifying $L[x]^n = L[x] \otimes_{K[x]} \cdots \otimes_{K[x]} L[x]$ with $L^n[x]$ and sending $f(x)$ to $f(0)$; this induces a mapping of Amitsur complexes, since $L^n[x]^* \to L^{n*}$. Finally, we have the induced homomorphism α'': $H^2(L[x]/K[x]) \to H^2(L/K)$. Now, by Theorems 2, 3, and 4, we have isomorphisms β: $H^2(L[x]/K[x]) \to \mathcal{B}(L[x]/K[x])$ and β'': $H^2(L/K) \to \mathcal{B}(L/K)$, and the first functorial property of the β's (§3, after Theorem 4) amounts to $\alpha\beta = \beta''\alpha''$.

THEOREM 7. *Let K be a field, L an extension field of finite degree over K* (*or even a commutative semisimple K algebra*), *and x an indeterminate over K. If K is perfect, or, more generally, if L is separable over K then the mapping*

(17) Note that $L[x] \otimes_{K[x]} K = L$.

α: $\mathfrak{B}(L[x]/K[x])\to\mathfrak{B}(L/K)$, given by $A\to A\otimes_{K[x]}K$, is an isomorphism. If K is imperfect, there is a field L of finite degree over K such that Ker $\alpha\neq 0$.

Proof. By the above discussion, it will be sufficient to prove the assertions for the homomorphism α'': $H^2(L[x]/K[x])\to H^2(L/K)$. Now, if L is separable, L^n is commutative and semisimple. The units in $L^n[x]$ are just the units of L^n (this is true because L^n is a direct sum of fields). Hence α' is an isomorphism of Amitsur complexes and α'' is an isomorphism of homology groups.

If K is imperfect, there exists an extension field $L=K(a)$ with a^p in K but not in K^p. We set

$$\theta = (1 \otimes a^{p-1} \otimes 1 - 1 \otimes 1 \otimes a^{p-1})(a \otimes 1 \otimes 1 - 1 \otimes a \otimes 1) \text{ in } L^3,$$

so that $\theta^p=0$, and define

$$t = \exp \theta x \text{ in } (L[x])^{3*}.$$

The same proof as that following Lemma 4.9 shows that t is a cocycle. We shall prove that t is not a coboundary.

Let D be the derivation of $L[x]$ defined by $D(a)=1$, $D(K[x])=0$, and define ${}_nD$ on $(L[x])^n$ by (4.1). Then, if t is a coboundary, $t=\epsilon_1(u)\epsilon_3(u)/\epsilon_2(u)$ for some $u\in L[x]^2$. Computing with logarithmic derivatives (cf. (4.12) and its proof), we obtain

$$(\epsilon_3 - \epsilon_2)({}_2Du/u) = {}_3Dt/t = ({}_3D\theta)x = (\epsilon_3 - \epsilon_2)(1 \otimes a^{p-1})x.$$

Thus ${}_2Du/u$ differs from $(1\otimes a^{p-1})x$ by an element of $\mathrm{Ker}(\epsilon_3-\epsilon_2)$ which, by Lemma 4.1, is $L[x]\otimes_{K[x]}1=(L\otimes_K 1)[x]$. Thus

$${}_2Du/u = (1 \otimes a^{p-1})x + \sum_{i=0}^{n} b_i x^i, \qquad b_i \text{ in } L \otimes_K 1.$$

Now we apply Lemma 4.4, (4.7) and the fact that $D^p=0$ to get

$${}_2D^{p-1}({}_2Du/u) + ({}_2Du/u)^p = 0.$$

On substituting for ${}_2Du/u$, this becomes

$$(5.1) \qquad \sum_{i=0}^{n} ({}_2D^{p-1}b_i)x^i + (1 \otimes a^{p(p-1)})x^p + \sum_{i=0}^{n} b_i^p x^{ip} = 0.$$

If the degree of ${}_2Du/u$ is $n>1$, then the left side of (5.1) has leading coefficient b_n^p which is $\neq 0$ since b_n is a nonzero element of the field $L\otimes_K 1$. Thus $n=1$ and (5.1) becomes

$$(b_0^p + {}_2D^{p-1}b_0) + {}_2D^{p-1}b_1 x + (a^{p(p-1)} \otimes 1 + b_1^p)x^p = 0.$$

Equating coefficients, we get $b_1=-a^{p-1}\otimes 1$ and ${}_2D^{p-1}b_1=0$ which are contradictory because $D^{p-1}(a^{p-1})=(p-1)!\neq 0$. Thus Theorem 7 is proved.

THEOREM 8 [5, *Theorem* 7.5]. *The homomorphism* α: $\mathcal{B}(K[x])\rightarrow\mathcal{B}(K)$ *has trivial kernel if and only if K is perfect.*

Proof. By Tsen's theorem ($\mathcal{B}(\overline{K}(x))=1$ where $\overline{K}$ is the algebraic closure of K) and a theorem of Auslander-Goldman ($\mathcal{B}(\overline{K}[x])\rightarrow\mathcal{B}(\overline{K}(x))$ is a monomorphism [5, Theorem 7.2]), we have $\mathcal{B}(\overline{K}[x])=1$. This means that if A is central separable over $K[x]$ then $A\otimes\overline{K}[x]$ is split; thus $\mathcal{B}(K[x])$ coincides with its subgroup $\mathcal{B}(\overline{K}[x]/K[x])$. By Lemma 3.16, the latter group is the union of all $\mathcal{B}(L[x]/K[x])$ taken over all fields L of finite degree over K. By Theorem 7, if K is perfect, then α has trivial kernel on every $\mathcal{B}(L[x]/K[x])$, hence also on $\mathcal{B}(K[x])$. If K is not perfect, Theorem 7 asserts that α has a nontrivial kernel in some $\mathcal{B}(L[x]/K[x])$, hence in $\mathcal{B}(K[x])$.

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